EXAMPLES OF ERGODIC DYNAMICAL SYSTEMS

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INTRODUCTION. Recently the author has established the following criterion for ergodicity of the measure-preserving flow on the torus[(1)].

Let Ω be a torus whose points can be represented by the coordinates (x, y), $0 \leq x, y < 2\pi$. Let S_t be a one-parameter stationary flow on Ω defined by

$$(1)\quad \begin{cases} \dfrac{dx}{dt} = X(x,y), \\ \dfrac{dy}{dt} = Y(x,y), \end{cases}$$

where X and Y are one-valued real functions on Ω having continuous first derivatives. If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0,$$

S_t is measure-preserving, or, in other words, differential equations (1) admit an integral invariant

$$\iint dx\, dy.$$

In this case, S_t is ergodic if and only if

1) X and Y have no common zero points, and

2) $\int_0^{2\pi}\int_0^{2\pi} X\, dx dy \neq 0$, $\int_0^{2\pi}\int_0^{2\pi} Y\, dx dy \neq 0$, and $\int_0^{2\pi}\int_0^{2\pi} X\, dx dy \Big/ \int_0^{2\pi}\int_0^{2\pi} Y\, dx dy$ is an irrational number.

This criterion can be immediately generalized to the case when S_t admits an integral invariant

$$\iint \rho(x,y)\, dx\, dy$$

where $\rho(x,y)$ is a positive definite function on Ω having continuous first derivatives. In such a case, we have

$$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y} = 0.$$

Hence by the similar discussion, we can easily show that S_t is ergodic if and only if

1) X and Y have no common zero points, and

2) $\int_0^{2\pi}\int_0^{2\pi} \rho X\, dx dy \neq 0$, $\int_0^{2\pi}\int_0^{2\pi} \rho Y\, dx dy \neq 0$, and $\int_0^{2\pi}\int_0^{2\pi} \rho X\, dx dy \Big/ \int_0^{2\pi}\int_0^{2\pi} \rho Y\, dx dy$ is an irrational number.

In this paper, we apply this criterion to several dynamical systems of two degrees of freedom and prove the existence of ergodic orbits.

EXAMPLE 1. Combination of two simple harmonic motions.

Consider the particle of unit mass on (x, y)-plane moving under the force $(-\alpha^2 x, -\beta^2 y)$. In this case, the particle generally describes a complicated path known as Lissajous' figure.

If we write

$$p_x = \frac{dx}{dt}, \quad p_y = \frac{dy}{dt},$$

the Hamiltonian of the system is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + \alpha^2 x^2 + \beta^2 y^2).$$

As is well known, this system admits two independent integrals

$$p_x^2 + \alpha^2 x^2 = c_1^2,$$
$$p_y^2 + \beta^2 y^2 = c_2^2,$$

where $c_1 > 0$, $c_2 > 0$ are integration constants. The integral surface defined by the above formulae is evidently homeomorphic to the torus. If we put

$$p_x = c_1 \cos\theta, \quad \alpha x = c_1 \sin\theta,$$
$$p_y = c_2 \cos\varphi, \quad \beta y = c_2 \sin\varphi,$$

the position of a point on this surface is represented by (θ, φ), $0 \leq \theta, \varphi < 2\pi$. Equations of motion are

$$\frac{d\theta}{dt} = \alpha, \quad \frac{d\varphi}{dt} = \beta.$$

Hence the system is ergodic if and