ON THE GROUP OF FORMAL ANALYTIC TRANSFORMATIONS

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1. Let K be an algebraic field. Under a (k-dimensional) formal analytic transformation[1] we mean a k-ple of integral formal power series in k variables $x_1, \ldots, x_k$ over K without constant terms. Let a and b be formal analytic transformations;

$$a: f_i(x) = f_i(x_1, \ldots, x_k) = \sum a^i_{j_1 \ldots j_k} x_1^{j_1} \ldots x_k^{j_k},$$

$$b: g_i(x) = g_i(x_1, \ldots, x_k) = \sum b^i_{j_1 \ldots j_k} x_1^{j_1} \ldots x_k^{j_k}.$$

$$(j_1 \geq 0, \ldots, j_k \geq 0, j_1 + \ldots + j_k \geq 1)$$

The product ab can be expressed as follows;

$$ab: h_i(x) = f_i(g_1(x), \ldots, g_k(x)),$$
$$(i=1, \ldots, k)$$

where it is to be noticed that the coefficients of ab can be determined formally as polynomials of those of a and b. The associativity of this multiplication is easy to verity and we obtain a semi-group F_k composed of all formal analytic transformations, whose identity is

$$e: e_i(x) = x_i. \quad (i=1, \ldots, k)$$

Next, letting correspond to any element a of F_k the linear part

$$La: f_i(x) = a^i_{1 0 \ldots 0} x_1 + \ldots + a^i_{0 \ldots 0 1} x_k,$$
$$(i=1, \ldots, k)$$

we have a linear representation of F_k;

$$(1) \qquad La \cdot Lb = L(ab).$$

Now let E_k be the group composed of all elements having inverses in F_k. E_k may be called the group of k-dimensional formal analytic transformations. An element a of F_k belongs to E_k if and only if La is a non-singular linear transformation. Now we define two subgroups of E_k in the following manner;

$$L_k = \{a : La = a\},$$
$$R_k = \{a : La = e\}.$$

Then (1) implies that R_k is an invariant subgroup such that

$$(2) \qquad E_k = R_k L_k, \quad R_k \cap L_k = e.$$

Now, let G be a group, and G, D(G), ..., $D_n(G) = D(D_{n-1}G)$, ... the descending series of subgroups of G, where D(G) denotes the commutator subgroup of G. When $\cap D_n(G) = e$, we shall call G solvable.

PROPOSITION 1. <u>R is a solvable group.</u>

Proof. Let

$$a: f_i = x_i + \sum_{n=2}^{\infty} A^i_n(x)$$
$$(i=1, \ldots, k)$$

be an element of R_k, where $A^i_n(x)$ denotes the homogeneous part of degree n. If a is not the identity, there exists $A^i_n(x) \neq 0$. The smallest number r such that there exists $A^i_r(x) \neq 0$ for some i is called the rank of a: r(a) = r. The rank of e is ∞.

Now let a and b be elements of R_k, of rank r and s respectively;

$$a: f_i = x_i + A^i_r + \text{higher terms},$$
$$b: g_i = x_i + B^i_s + \text{higher terms}.$$
$$(i=1, \ldots, k)$$

Then clearly we have

(3) ab:

$$h_i = \begin{cases} x_i + (A^i_r + B^i_r) + \text{higher terms}, \\ x_i + A^i_r \qquad + \text{higher terms}, \\ x_i + B^i_s \qquad + \text{higher terms}, \end{cases}$$

$$(i=1, \ldots, k)$$

according as r=s, r<s, or r>s respectively. Hence in the expressions of ab and ba the terms of degree Min(r,s) concide, and this readily leads to the following inequality;

$$(4) \qquad r(aba^{-1}b^{-1}) > \mathrm{Min}(r(a), r(b)),$$

which is valid except for a=b=e.

Let R^t_k be the subset of R_k composed of all elements of rank at least t(t≥2). By (3) R^t_k is a subgroup, and we have that $\cap R^t_k = e$. On the other hand we can conclude from (4) that

$$D(R_k) \subseteq R^3_k, \quad D_2(R_k) \subseteq R^4_k, \ \ldots,$$

whence R_k is solvable.

2. In this section we consider the case where K is the field of complex (or real) numbers. Then we can introduce a topology (the so-called weak topology) in F, namely the sequence $\{a(n)\}$;