ON AN ASYMPTOTIC PROPERTY OF A GAP SEQUENCE

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Let $f(t)$ be a real measurable function of a real variable t, and $\{\lambda_k\}$ be an increasing positive sequence with certain gap conditions. Asymptotic properties of the sequence of functions $f(\lambda_k t)$, such as the asymptotic distribution of their partial sums and almost everywhere convergence or divergence of the series $\sum c_k f(\lambda_k t)$, c_k constant coefficients, have been discussed by M.Kac,[1] R.Fortet,[2] R.Salem and A.Zygmund,[3] R.Fortet and J.Ferrand,[4] and T.Kawata.[5] The object of this note is to prove the following theorem which corresponds to the law of the iterated logarithm in the theory of probability. The proof given here depends on Kac's method of approximating a gap sequence by a set of independent functions.

<u>Theorem. Let $f(t)$, $0\le t\le 1$, be a periodic continuous function which has the vanishing mean $\int_0^1 f(t)\,dt=0$ and satisfies a Lipschitz condition of order α, $\alpha>0$. Then we have</u>

$$\overline{\lim_{n\to\infty}}\frac{f(2t)+f(2^2t)+\cdots+f(2^nt)}{\sqrt{2n\log\log n}}=\sigma$$

<u>for almost all</u> t, where

$$\sigma^2=\lim_{n\to\infty}n^{-1}\int_0^1\{f(2t)+\cdots+f(2^nt)\}^2dt.\ ^{6)}$$

<u>Proof.</u> Let us put $\lambda(x)=[c\log x]$, where $c=2/\alpha\log 2$ and $[a]$ denotes the integral part of a, and define non-negative sequences of integers $\{d_\nu\}$, $\{p_\nu\}$, $\{m_\nu\}$, $\{n_\nu\}$ in the following way. Choose an integer N so large, that we have

$$c\log\{2\nu(\log\nu)^2\}<[(\log\nu)^2]\qquad\text{for } \nu\ge N$$

and let $1\le d_1\le\cdots\le d_{N-1}\le[(\log N)^2]$ but d_ν $(1\le\nu\le N-1)$ otherwise arbitrary; $d_\nu=[(\log\nu)^2]$ for $\nu\ge N$; $p_\nu=d_\nu$ for $1\le\nu\le N-1$;

$$p_\nu=\lambda(d_1+\cdots+d_\nu+p_1+\cdots+p_{\nu-1})$$

for $\nu\ge N$, and put

$$(1)\qquad \begin{aligned}m_\nu&=d_1+\cdots+d_\nu+p_1+\cdots+p_{\nu-1}\\ n_\nu&=m_\nu+\lambda(m_\nu)\end{aligned}$$

Then by the above choice of N and definition of p_ν

$$(2)\qquad p_N\le\lambda(2Nd_N)\le c\log\{2N(\log N)^2\}\le d_N$$

$$(3)\qquad p_\nu>\lambda(d_1+\cdots+d_\nu)\qquad\text{for } \nu\ge 1,$$

and hence, starting from (2), we can inductively show that we have

$$(4)\qquad p_\nu\le d_\nu\qquad\text{for all } \nu\ge 1.$$

Now, for any positive sequences $\{a_\nu\}$ and $\{b_\nu\}$ let us agree to write $a_\nu\sim b_\nu$, when $a_\nu/b_\nu\to 1$, as $\nu\to\infty$. Then, taking into account of (3) and (4) we obtain

$$(5)\qquad p_\nu\sim c\log\nu,\quad d_\nu\sim(\log\nu)^2,\quad m_\nu\sim n_\nu\sim\nu(\log\nu)^2.$$

Consider the dyadic expansions of all real numbers t, $0\le t\le 1$,

$$t=\frac{\varphi_1(t)}{2}+\frac{\varphi_2(t)}{2^2}+\cdots,\quad \varphi_i(t)=\frac{1+r_i(t)}{2},$$

where by $r_i(t)$ we denote Rademacher's system of independent functions, and put

$$\theta_m(t)=\frac{\varphi_m(t)}{2}+\cdots+\frac{\varphi_{m+\lambda(m)}(t)}{2^{1+\lambda(m)}},$$

$$\mu_\nu=\int_0^1 f(\theta_\nu(t))\,dt,\qquad x_\nu=x_\nu(t)=f(\theta_\nu(t))-\mu_\nu.$$

$$(6)\qquad S_\nu=\sum_{p=n_{\nu-1}+1}^{m_\nu}x_p,\quad T_\nu=\sum_{p=m_\nu+1}^{n_\nu}x_p,$$

$$\sigma_\nu^2=\int_0^1 S_\nu^2\,dt,\qquad \tau_\nu^2=\int_0^1 T_\nu^2\,dt$$

Then by the periodicity and Lipschitz condition imposed on $f(t)$ we obtain

$$(7)\qquad f(2^\nu t)-f(\theta_{\nu+1}(t))=O(2^{-\alpha\lambda(\nu)})=O(\nu^{-2}),$$

and therefore

$$(8)\qquad \mu_\nu=\int_0^1 f(2^{\nu-1}t)\,dt+\int_0^1\{f(\theta_\nu(t))-f(2^{\nu-1}t)\}\,dt=O(\nu^{-2}).$$

(6), (7) and (8) give us

$$(9)\qquad \sigma_\nu^2=\int_0^1\{f(t)+\cdots+f(2^{d_\nu}t)\}^2dt+o(1)=d_\nu\sigma^2(1+o(1)),$$

and similarly

$$(10)\qquad \tau_\nu^2=p_\nu\sigma^2(1+o(1)).$$

Since, as is seen from (6), S_ν depends only on $r_i(t)$, $n_{\nu-1}+1\le i\le n_\nu$, $\{S_\nu\}$ is a set of independent functions, each with common mean value 0, and the same is true of $\{T_\nu\}$. Hence, if we write

$$(11)\qquad \sum_{p=1}^{n_k}x_p=\sum_{\nu=1}^{k}S_\nu+\sum_{\nu=1}^{k}T_\nu,$$