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THE CARATHÉODORY METRIC IN PLANE DOMAINS

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Abstract. Let $D \notin O_{AB}$ be a plane domain and let $C_D(z)$ be its analytic capacity at $z \in D$. Let $\mathcal{K}_D(z)$ be the curvature of the Carathéodory metric $C_D(z)|dz|$. We show that $\mathcal{K}_D(z) < -4$ if the Ahlfors function of D with respect to z has a zero other than z. For finite D, $\mathcal{K}_D(z) \leqq -4$ and equality holds if and only if D is simply connected. As a corollary we obtain a result proved first by Suita, namely, that $\mathcal{K}_D(z) \leqq -4$ if $D \notin O_{AB}$. Several other properties related to the Carathéodory metric are proven.

§ 1. Introduction.

Let $D \notin O_{AB}$ be a plane domain and let $C_D(z)$ be its analytic capacity at $z \in D$. Let $\mathcal{K}_D(z)$ be the curvature of the Carathéodory metric $C_D(z)|dz|$. This paper is divided into three main parts. In the first part we study the curvature $\mathcal{K}_D(z)$ and its relation with the curvatures of the Bergman metrics (Theorem 1). As a corollary, we show that the boundary value of $\mathcal{K}_D(z)$ is -4 if D is bounded by C^2 curves.

The second part of this paper is devoted to the discussion of the boundedness of $\mathcal{K}_D(z)$ by -4. Suita [8], using the notion of supporting metrics, has shown that $\mathcal{K}_D(z) \leqq -4$. However, this method of proof seems not to apply for deciding when $\mathcal{K}_D(z) = -4$ occurs. After submitting this paper for publication the author's attention was drawn to the existence of another recent paper of Suita [9] wherein a different method of proof is provided to settle the above question when D is finite. This method is similar to the one we employed in our Theorem 2 and it is based on the "method of minimum integral" with respect to the Szegö kernel function. In fact, with similar methods and without being aware of [9] we were able to extend some of the present results to higher order curvatures [2]. This part is revised accordingly. The author is most indebted to Professor N. Suita for his valuable comments and suggestions, and, in particular, for pointing out the possibility of sharpening our assertions to the form embodied in Theorem 3 of this paper.

Finally, in the third part we study in some detail the curvature $\mathcal{K}_D(z)$ for

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