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FABER'S POLYNOMIALS.

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§1. Fundamental Identities.

The following method in §1 can be proceded verbatin for more general and even for multiply-connected domains, but in this Note we suppose the boundary of domain is the unit circle in order to apply our results for the coefficient problem.

Let $g(z)$ be a meromorphic, schlicht and non-vanishing function in the exterior of the unit circle $|z| > 1$, and whose Laurent expansion about the point at infinity is of the form

$$(1) \qquad g(z) = z + \sum_{\nu=0}^{\infty} \frac{c_\nu}{z^\nu}.$$

Then $f(z) \equiv 1/g(1/z)$ is regular and schlicht in the unit circle $|z| < 1$ and, about the origin, it can be expanded in the form

$$(2) \qquad f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu.$$

Let $P_n(z)$ $(n = 1, 2, \cdots)$ be polynomial of z of degree n, which satisfies the condition

$$(3) \qquad P_n(g(z)) = z^n + \sum_{\nu=1}^{\infty} \frac{\alpha_\nu^{(n)}}{z^\nu}.$$

Then, $P_n(z)$ is called the "Faber's polynomial" of degree n with respect to $g(z)$.[1] By means of the Cauchy's integral formula, we have

$$(4) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{P_n(\zeta)}{\zeta - w}\, d\zeta,$$

where w is an arbitrary point in the circle $|\zeta| < r$. Making the change of variable $\zeta = g(z)$, we get, for sufficiently large r,

$$(5) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d\lg\, (g(z) - w).$$

On the other hand, we can easily prove

$$(6) \qquad \begin{aligned} 0 &= \frac{1}{2\pi i}\int_{|z|=r} \frac{P_n(g(z))}{z}\, dz \\ &= \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d\lg z, \end{aligned}$$

by virtue of (3). Hence, substracting (6) from (5), we have

$$(7) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d\lg \frac{g(z)-w}{z}$$

Now, putting

$$(8) \qquad \lg \frac{g(z)-w}{z} = -\sum_{\nu=1}^{\infty} \frac{Q_\nu(w)}{\nu}\frac{1}{z^\nu}$$

and substituting (8) into (7), we obtain

$$(9) \qquad P_n(w) = Q_n(w).$$

Since (9) holds for infinitely many values of w if we take a sufficiently large r, also does (9) hold good identically. After all, we have the following fundamental relation:[2]

$$(10) \qquad \lg \frac{g(z)-w}{z} = -\sum_{\nu=1}^{\infty} \frac{P_\nu(w)}{\nu}\frac{1}{z^\nu}$$

for an arbitrary w, the logarithm always denoting the branch which vanishes for $w = 0$ and $z = \infty$.

Putting $\zeta = 1/z$, $g(z) = 1/f(\zeta)$, and comparing the coefficients of both sides of (10), we have

$$(11) \qquad P_n(z) = n\sum_{\mu=1}^{n}\Big(\sum_{n_1+\cdots+n_\mu=n} a_{n_1}\cdots a_{n_\mu}\Big)\frac{z^\mu}{\mu} + P_n(0),$$

and in particular $P_n(0) = n a_n$. Differentiating (10) with respect to z and making use of the same reason as above, we obtain

$$(12) \qquad \begin{gathered} P_1(z) = z - c_0, \\ P_{n+1}(z) + (c_0 - z)P_n(z) + \sum_{\mu=1}^{n-1} c_\mu P_{n-\mu}(z) + (n+1)c_n = 0 \\ (n = 2, 3, \quad) \end{gathered}$$

§2. Some Applications to the Distortion Theorems.

Putting

$$(13) \qquad F(z) = \begin{cases} f(z)/z & (z \neq 0), \\ 1 & (z = 0), \end{cases}$$

we get, from (10),

$$(14) \qquad \lg F(z) = \sum_{\nu=1}^{\infty} \frac{P_\nu(0)}{\nu} z^\nu.$$

If we consider a family of schlicht