NOTE ON GROUPS OF AUTOMORPHISMS.

By Noboru ITÔ and Masayoshi NAGATA

(Communicated by S. Ikehara)

In this paper, we shall denote by e the identity element of a group, while we denote by E the identity group. And we shall denote by $A(G)$ the group of automorphisms of a group G.

Definition 1. Let G_0 be a group whose center is E. Consider the chain

$$G_0 < G_1 < \cdots < G_n < \cdots$$

satisfying the following conditions;

(1) if there exists G_i $(i \geq 1)$, $G_i = A(G_{i-1})$;

(2) when G_i exists and G_{i+1} does not exist, $A(G_i) = G_i$.

Then we shall call this chain the tower of G_0. If there exists a last term, say G_n, of this chain, we shall say that the length of this tower is n, and denote G_n by $\overline{G}_0$. Otherwise we shall say that the length of this tower is infinite.

Remark: Wielandt has proved that a tower of any finite group has a finite length (cf. Math. Zeitschr. 45 (1939)).

Definition 2. We shall say that a group G is complete if (i) the center of G is E and (ii) $\overline{G} = G$ (i.e. $A(G) = G$).

§1. Groups of dihedral type.

Definition 3. We shall say that G is of dihedral type, if (i) G has a normal subgroup U such that $(G : U) = 2$, (ii) G has an element x of order 2 such that $xax = a^{-1}$ for every $a \in U$ and (iii) $G = U.\{x\}$.

Remark: From this definition, it follows easily that U is an abelian group.

Lemma 1. Let G be a group of dihedral type. Then the center of G is E if and only if U (with the notation introduced in definition 3) has no element of order even.

Proof. If U has an element a of order 2, a is contained in the center of G. In the other case, it is clear that the center of G is E.

Lemma 2. Let G be a group of dihedral type with minimal condition on normal subgroups. Then the order of G is finite.

Proof. This follows readily from that every subgroup of U (in definition 3) is a normal subgroup of G.

From now on we shall consider only groups with minimal condition on normal subgroups if contrary is not expressed.

Proposition 1. Let G be a group of dihedral type. Then G is complete if and only if G is the symmetric group of degree 3 (we shall denote this by S_3). (Cf. example 2, §4)

Proof. If U (with notation in definition 3) has an element of order k $(k > 3)$ then G has an outer automorphism σ such as $a^\sigma = a^2$ for every $a \in U$ and $x^\sigma = x$ (with notations in definition 3). Therefore, if $(U : E) > 3$, U is of the type $(3, 3, \ldots, 3)$. Then G has outer automorphisms; for instance, those which permute the basis of U and leave x fixed. So G must be S_3. On the other hand, it is clear that S_3 is complete.

§2. On $A(G \times G)$.

Proposition 2. Let G be a complete, directly indecompisable group. Then $A(G \times G) = (G \times G)\{y\}$ where $y^2 = e$, $y(a, b)y = (b, a)$ for every $(a, b) \in G \times G$. Furthermore $A(G \times G)$ is directly indecomposable.

Proof. Let σ be an automorphism of $G \times G$. We set $H_1 = \{(a, e); (ab, c) = (ba, c)$ for every $(b,c) \in (G \times E)^\sigma\}$ and $H_2 = \{(e, a); (b, ac) = (b, ca)$ for every $(b, c) \in (G \times E)^\sigma\}$. Then we have $(E \times G)^\sigma = H_1 \times H_2$. This implies that $(E \times G)^\sigma = E \times G$ or $G \times E$, because G is directly indecomposable. If we observe that G is complete, we have the first part of proposition 2.

Assume now that $A(G \times G) = M \times N$ where $M \neq E$ $N \neq E$, and set

$K_1 = M \cap (G \times G)$, $K_2 = N \cap (G \times G)$. Then we have $(M : K_1) = 2$ or $M = K_1$, $(N : K_2) = 2$ or $N = K_2$. If $K_2 = E$,