CONFORMAL MAPPING OF POLYGONAL DOMAINS.

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1. Introduction. A function which maps a circular disc or a half-plane conformally onto a rectilinear polygon is, as is well known, given by Schwarz-Christoffel formula. Let $w=f(z)$ be such a function, and let the interior angle at vertex $f(a_\mu)$ ($\mu=1,\cdots,m$) of the image-polygon, having m vertices, be denoted by $\alpha_\mu\pi$, the formula may be written in the form:

$$(1.1)\quad f(z)=C\int^z\prod_{\mu=1}^{m}(a_\mu-z)^{\alpha_\mu-1}dz+C',$$

where C and C' are both constants depending only on position and magnitude of the image-polygon.

The present author[1] has previously shown that this formula can be generalized to the case of analogous mapping of doubly-connected domains. We may adopt, as a standard doubly-connected basic domain, an annular domain $q<|z|<1$, $-\lg q$ being a uniquely determined conformal invariant, i.e. the so-called modulus of given polygonal ring domain. Let the boundary components corresponding to circumferences $|z|=1$ and $|z|=q$ be rectilinear polygons with m and n vertices respectively. Let further $\alpha_\mu\pi$ and $\beta_\nu\pi$ denote the interior angles (with respect to each boundary polygon itself) at vertices $f(e^{i\varphi_\mu})$ and $f(qe^{i\psi_\nu})$ respectively. The mapping function $w=f(z)$ is then expressed in the form:

$$(1.2)\quad f(z)=C\int^z z^{ic^*-1}\Big\{\prod_{\mu=1}^{m}\sigma(i\lg z+\varphi_\mu)^{\alpha_\mu-1}\div\prod_{\nu=1}^{n}\sigma_3(i\lg z+\psi_\nu)^{\beta_\nu-1}\Big\}dz+C',$$

where the sigma-functions are those of Weierstrass with primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$ and the constant c^* is given by

$$(1.3)\quad c^*=\frac{\eta_1}{\pi}\Big(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu-\sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu\Big);$$

the constants C and C' having similar meanings as before. It can, moreover, be shown that the Schwarz-Christoffel formula (1.1), for basic domain $|z|<1$, may be regarded as being a limiting case of (1.2) when $q\longrightarrow 0$.

On the other hand, any function $w=f(z)$ which maps a circular disc or a half-plane conformally onto the interior of a circular polygonal domain, i.e. the interior of a polygon having circular arcs as its sides, is linear-polymorphic. A differential equation of the third order of the form:

$$(1.4)\quad \{f(z),\ z\}=R(z)$$

holds good always for such a function $f(z)$. The left member of this equation denotes, as usual, Schwarzian derivative of $f(z)$ with respect to z, i.e.

$$\{f(z),z\}\equiv\frac{d^2}{dz^2}\lg\frac{df(z)}{dz}-\frac{1}{2}\Big(\frac{d}{dz}\lg\frac{df}{dz}\Big)^2=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2,$$

and $R(z)$ is a rational function which possesses, as poles of order at most two, the points a_μ ($\mu=1,\cdots,m$) corresponding to the vertices of image-polygon. More precisely, if we denote by $\alpha_\mu\pi$ the interior angle at $f(a_\mu)$ of the image-polygon, we have, at the pole in question, the relation:

$$(1.5)\quad \lim_{z\to a_\mu}(z-a_\mu)^2R(z)=\frac{1-\alpha_\mu^2}{2}.$$

The above mentioned results (1.1) and (1.4) are usually derived by making use of analytic continuability of mapping function, that is, by performing successive inversions with respect to boundary arcs. But the author of this paper[2] previously pointed out that Schwarz-Christoffel formula (1.1) can be deduced immediately from Poisson integral representation of functions analytic in a circular disc. He[1] also has derived the formula (1.2) by means of Villat's integral representation of functions analytic in an annular domain. It will, however, be shown that the formula (1.2) can also be derived by the classical method without particular difficulty.

We can, on the other hand, consider the problem of generalization of (1.4) corresponding to that of (1.1) to (1.2).