DEGREE FORMULAS FOR A TOPOLOGICAL INVARIANT OF BIFURCATIONS OF FUNCTION-GERMS

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1. Introduction

Let $F = (f_1, \ldots, f_k) : (K^n, 0) \rightarrow (K^k, 0)$, with $1 \leq k < n$ and $K = C$ or $K = R$, be an analytic map-germ with an isolated singularity at the origin. Let $g : (K^n, 0) \rightarrow (K, 0)$ be an analytic function-germ. We are interested in computing topological invariants associated to the mappings $F$ and $(F, g)$.

Let $B_\varepsilon \subset K^n$ be a small ball centered at the origin and let $\delta \in K^k$ be a small regular value of $F$. The Milnor fiber of $F$ is $F^{-1}(\delta) \cap B_\varepsilon$. If $k = 1$, Milnor [Mi2] proved that $F^{-1}(\delta) \cap B_\varepsilon$ has the homotopy type of a bouquet of $\mu (n - 1)$-spheres where

$$
\mu = \dim_C \mathcal{O}_{C^n, 0} \left/ \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) \right.
$$

These results were extended to the case $1 < k < n$ by Hamm [Ha], who proved that the Milnor fiber has the homotopy type of a bouquet of $\mu (n - k)$-spheres, and by Lé [Le] and Greuel [Gr] who obtained the formula

$$
\mu(F') + \mu(F) = \dim_C \mathcal{O}_{C^n, 0} / I,
$$

where $F' = (f_1, \ldots, f_{k-1})$ and $I$ is the ideal generated by $f_1, \ldots, f_{k-1}$ and all the $k \times k$ minors $\partial(f_1, \ldots, f_k)/\partial(x_1, \ldots, x_n)$.

For the real case, it is difficult to give such precise informations about the topology of the Milnor fiber. Nevertheless it is possible to compute some Euler characteristics. For instance, if $k = 1$, the Khimshiasvili’s formula ([Ar], [Fu], [Kh], [Wa]) states that

$$
\chi(F^{-1}(\delta) \cap B_\varepsilon) = 1 - \text{sign}(-\delta)^n \deg_0 \nabla F,
$$

where $\chi(-)$ denotes the Euler-Poincaré characteristic and $\deg_0 \nabla F$ is the topological degree of the gradient of $F$ at the origin.

It is also possible to compute the following difference

$$
D_{\delta, x} = \chi(F^{-1}(\delta) \cap \{ g \geq x \} \cap B_\varepsilon) - \chi(F^{-1}(\delta) \cap \{ g \leq x \} \cap B_\varepsilon),
$$

for $(\delta, x)$ a suitable regular value of $(F, g)$.