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COMPLEX CONTACT MANIFOLDS AND HYPERKÄHLER GEOMETRY*

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Abstract

We investigate the properties of a certain class of hermitian metrics on complex contact manifolds with global complex contact forms. These are called complex Sasakian metrics. In the main theorem, we prove a relationship between complex Sasakian geometry and hyperkähler geometry. We then compare these complex contact manifolds to the more famous twistor space examples.

1. Introduction

In times past, there have been many interesting and useful correspondences proven between real contact and almost complex geometries. For example, Hatakeyama proved in [12] that every Sasakian manifold locally fibres over a Kähler manifold. Also, the Boothby-Wang Fibration gives a construction by which an S^1-bundle with a real contact structure is formed over a symplectic manifold [5]. Furthermore, Hatakeyama's result can be generalized to make the Boothby-Wang fibration a Riemannian submersion of associated metrics.

A great deal work has also been done to find similar correspondences dealing with complex contact and quaternionic structures on manifolds. However, a large obstacle has been finding a proper quaternionic context to work with. An almost quaternionic structure is defined easily enough; it is simply a metric g together with a three-dimensional bundle E of endomorphisms with local bases $\{A, B, C = AB\}$ of local anti-commuting almost complex structures such that g is hermitian with respect to each endomorphism.

What is the analogue for g being Kähler? If we just insist that each almost complex structure of E be parallel, then the quaternionic projective space HP^n would be excluded (see [3]). To remedy this, we define an almost quaternionic manifold (M, g, E) to be *quaternionic-Kähler*, if $\nabla E \subset T^*M \otimes E$. Under these circumstances, g is Einstein, and so we can categorize these manifolds into three categories by the sign of the scalar curvature: zero, positive, or negative.

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