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ON THE EXISTENCE OF CERTAIN QUADRATIC DIFFERENTIALS ON FOUR TIMES PUNCTURED SPHERES AND ONCE PUNCTURED TORI

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Let R be a Riemann surface. Let $\{\gamma_j\}_{j=1}^{p}$ be a set of homotopically nontrivial Jordan curves on R which are mutually disjoint and belong to different free homotopy classes. We consider the following problem.

PROBLEM. Find conditions of non-negative numbers $l_1, \dots, l_p$ such that there exists the holomorphic quadratic differential φ with closed trajectories on R which has following properties:

(a) Each of closed trajectories of φ is homotopic to one of the curves $\{\gamma_j\}_{j=1}^{p}$.

(b) For any $j = 1, \dots, p$, φ has closed trajectories homotopic to γ_j.

(c) For $j = 1, \dots, p$, the φ-length of closed trajectories homotopic to γ_j is equal to l_j.

In this paper, we shall give answers for this problem in the case where R is either a four times punctured sphere or a once punctured torus (see Sections 3 and 5). An essential tool in obtaining our results is the deformation space of Riemann surfaces with nodes due to Bers.

This problem is related to the following Strebel's result (cf. Theorem 23.5 in [18, p. 150]).

THEOREM. *Given a Riemann surface R with marked points $P_j, j = 1, \dots, p$ $p \geq 2$ and $\dot{R} = R \setminus \{P_j\}$ not the twice punctured sphere. We consider the quadratic differentials φ on $\dot{R}$ with closed trajectories the characteristic ring domains of which are punctured discs R_j, with punctures P_j. Then, the lengths $a_j > 0$ of the closed trajectories α_j around the P_j can be prescribed arbitrarily. The solution φ is uniquely determined.*

Strebel proved this result by using the convexity of the surface of reduced moduli (cf. [18, p. 148]). This theorem implies that our problem is solved in the case where every γ_j is homotopic to a small loop around a puncture. In this case, constants $l_1, \dots, l_p$ are prescribed arbitrarily. Therefore we only consider

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