T. FUKAYA
KODAI MATH. J.
22 (1999), 140–151

ON HASSE ZETA FUNCTIONS OF GROUP ALGEBRAS OF ALMOST NILPOTENT GROUPS

TAKAKO FUKAYA

1. Introduction

1.1. In the paper [Fu], we generalized the Hasse zeta functions $\zeta_A(s)$ of commutative finitely generated rings A over the ring $\boldsymbol{Z}$ of integers, to non-commutative rings (concerning the definition, see below), and computed the zeta functions of some rings. The examples suggested a strong relationship between the convergence of the Hasse zeta functions and the Gelfand-Kirillov dimensions of the rings (concerning the definition of the Gelfand-Kirillov dimension, see section 2). So in [Fu], we conjectured the relationship between them (see section 2). In the present paper, we consider this conjecture in the case of group rings.

For a (not necessarily commutative) finitely generated ring A over $\boldsymbol{Z}$, in [Fu] we defined the Hasse zeta function $\zeta_A(s)$ of A by

$$\zeta_A(s) = \prod_M (1 - N(M)^{-s})^{-1}$$

where M runs over the isomorphism classes of finite simple A-modules and $N(M) = \sharp\mathrm{End}_A(M)$.

We say $\zeta_A(s)$ converges if there exists a real number d such that the product defining $\zeta_A(s)$ absolutely converges when $\mathrm{Re}(s) > d$, and we say $\zeta_A(s)$ diverges otherwise. In fact, the function $\zeta_A(s)$ diverges for some rings A.

The purpose of this paper is to prove

THEOREM 1.2. *Let G be a finitely generated group which has a nilpotent subgroup of finite index, and let R be a finitely generated commutative ring over $\boldsymbol{Z}$. Let A be the group ring $R[G]$. Then the function $\zeta_A(s)$ converges.*

We call a group which has nilpotent subgroup of finite index, an almost nilpotent group.

By a theorem of Gromov [Gr], for a finitely generated group G and for a field k, the Gelfand-Kirillov dimension of $k[G]$ (concerning the definition, see section 2) is finite if and only if of G is almost nilpotent. So Theorem 1.2 implies the following corollary (see also Corollary 2.6).

Received June 23, 1998.