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TOPOLOGY OF COMPLEX POLYNOMIALS VIA POLAR CURVES

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1. The main results

The use of the local polar varieties in the study of singular spaces is already a classical subject, see Lê-Teissier [LT] and the references therein.

In this note we consider the global polar curves associated with an affine smooth hypersurface F in $\boldsymbol{C}^n$. Instead of considering the higher dimensional polar varieties associated with F, we choose to look at the polar curves for the various generic linear sections of F. This approach is motivated by our use of classical dual varieties and also by our main interest in numerical invariants describing the topology of F in terms of these family of polar curves.

More precisely, let $f \in \boldsymbol{C}[x_1, \dots, x_n]$ be a polynomial and assume that the fiber $F_t = f^{-1}(t)$ is smooth and connected. Our main result computes the Euler characteristic $\chi(F_t)$ of the hypersurface F_t in terms of the polar invariants of the intersections $F_t \cap E^k$, where E^k is a general linear subspace in $\boldsymbol{C}^n$ of codimension k, for $k = 0, 1, \dots, n-1$.

First we define these polar invariants. For any hyperplane

$$H : h = 0 \textit{ where } h(x) = h_0 + h_1 x_1 + \cdots + h_n x_n$$

we define the corresponding polar variety Γ_H to be the union of the irreducible components of the variety

$$\{x \in \boldsymbol{C}^n \mid \operatorname{rank}(df(x), dh(x)) = 1\}$$

which are not contained in the critical set $S(f) = \{x \in \boldsymbol{C}^n \mid df(x) = 0\}$ of f.

Note that Γ_H depends only on the direction $H^d = (h_1 : \cdots : h_n) \in \boldsymbol{P}^{n-1}$ of the hyperplane H.

LEMMA 1. *For a generic hyperplane H we have the following properties.*

(i) *The polar variety Γ_H is either empty or a curve, i.e. each irreducible component of Γ_H has dimension* 1.

(ii) $\dim(F_t \cap \Gamma_H) \leq 0$ *and the intersection multiplicity (F_t, Γ_H) is independent of H.*

(iii) *The multiplicity (F_t, Γ_H) is equal to the number of tangent hyperplanes to F_t parallel to the hyperplane H. For each such tangent hyperplane H_a, the*

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