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MODULI OF RING DOMAINS OBTAINED BY A CONFORMAL WELDING

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Abstract

We are concerned with ring domains which are conformally welded along a pair of opposite sides of a square. Oikawa studied moduli of these ring domains and left some problems. We shall answer one of these open problems.

1. Introduction

Welding of polygons and the type of Riemann surfaces were considered by Nevanlinna, Oikawa and others (cf. [3], [4]). We are concerned with the relation of weldings and the moduli of Riemann surfaces. Oikawa studied this subject and got some results which he didn't publish (cf. [5]). We follow him. A square in the complex plane can be conformally welded into various ring domains by a specific kind of identification of a pair of opposite sides. We consider the range of these moduli. Oikawa gave an estimate for the range of these moduli and asked whether it is the best possible or not. We shall show a certain identification which give conformally welded ring domains with arbitrary small moduli. In addition, we shall show that the moduli of ring domains conformally welded by an unnatural identification never meet to a neighborhood of the module of ring domains conformally welded by the natural identification.

Consider the square $Q=\{x+iy : 0<x<1, 0<y<1\}$ in the complex plane, and put

$$L_+=\{x+iy : 0<x<1, y=1\}, \qquad L_-=\{x+iy : 0<x<1, y=0\}.$$

Let $\phi_0(x+i)=x(0<x<1)$ and $\phi : L_+\to L_-$ be a homeomorphism such that $\phi\circ\phi_0^{-1}(x)$ is strictly increasing. Let G be a ring domain and C be a Jordan curve in G joining two boundary components of G. Let f be a continuous mapping from $Q\cup L_+\cup L_-$ onto G. We say the triple (G, C, f) is a conformal welding obtained by ϕ if f is conformal in Q, $f\circ\phi=f$ on L_+, and $f(L_+)=f(L_-)=C$. And we call ϕ a welding function. We say a conformal welding by ϕ is unique, if, for any two conformal weldings $(G_i, C_i, f_i)_{i=1,2}$ obtained by ϕ, $f_2\circ f_1^{-1}$ is a con-

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