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TOTALLY GEODESIC SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS AND CURVATURE-INVARIANT SUBSPACES

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1. Introduction

An isometric immersion $\varphi: S \to M$ of a Riemannian manifold S into another Riemannian manifold M is called *totally geodesic* if the geodesics in S are carried into geodesics in M. We call such a pair (S, φ) a totally geodesic submanifold of M. Nevertheless, identifying S and the image $\varphi(S)$, we often call the subset $\varphi(S)$ in M a totally geodesic submanifold. Local problems are generally discussed in such a way. Among submanifolds of a Riemannian manifold, totally geodesic ones are fundamental.

Totally geodesic submanifolds of Riemannian symmetric spaces have been well investigated and it has been shown that they have beautiful and fruitful properties. In particular, due to the (M_+, M_-)-theory by B.Y. Chen and T. Nagano ([3]) this subject has made great progress. The author has a wish to understand well totally geodesic submanifolds of "general" Riemannian manifolds and obtained a few results in this paper.

We are concerned with three problems in this paper.

PROBLEM 1. *For a given subspace V in a tangent space T_pM, find good (or practical) criteria to conclude that there is a totally geodesic submanifold through p whose tangent space at p is V.*

For this problem, we recall a theorem of E. Cartan is section 2 (Theorem 2.1 in this paper), which becomes a remarkable criterion if M is a Riemannian symmetric space. That is, there is a totally geodesic submanifold whose tangent space is V if and only if V is a curvature-invariant subspace with respect to the Riemannian curvature tensor R, i.e.,

$$R(x, y)z \in V \quad \text{for any } x, y, z \in V. \tag{1.1}$$

Is there such a criterion as above for a wider class of Riemannian manifolds? For example, for homogeneous Riemannian manifolds does there exist a finite number d such that a condition similar to (1.1) for $R, \nabla R, \cdots, \nabla^d R$ implies the existence of totally geodesic submanifolds? In this paper we will show this for

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