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A SURVEY ON SINGULAR SOLUTIONS OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This is a summary of the papers [9,10,11,12,13,15], in which we mainly studied singular solutions of first order differential equations.

0. Introduction

How should we define the notion of singular solutions for differential equations? In [9,10,11,12,13,15] we have considered this philosophical and stimulating problem for first order differential equations. The first example of singular solutions was discovered by Brook Tayler about 290 years ago (cf. [7]). 20 years after that Alex Claude Clairaut studied a class of equations which have singular solutions [2]. This equation is called the Clairaut equation now: $y = x \cdot \frac{dy}{dx} + f(\frac{dy}{dx})$. It has a quite beautiful geometric structure as follows : There exists a "general solution" that consists of lines ; $y = t \cdot x + f(t)$, where t is a parameter and the singular solution is the envelope of such a family. In classical treatises of equations (Carathéodory [1] , Courant-Hilbert [3], Forsyth [4] [5], Ince [7], Petrovski [17]) the discussions of equations with singular solutions are informal. In these, a "general solution" of a differential equation is defined to be an one-parameter family of solutions and a "singular solution" is a solution which is not contained in the "general solution". However, this definition of singular solutions is very confused as the following example shows:

Example 0.1. Consider the equation $y = 2p \cdot x - p^2$, where $p = \frac{dy}{dx}$. In [7] the "general solution" is given by

$$\begin{cases} x = \frac{c}{p^2} + \frac{2}{3}p \\ y = 2p \cdot x - p^2, \end{cases}$$

where c is a parameter. It is clear that $y = 0$ is also a solution, but it is not contained in the "general solution". Then $y = 0$ must be the "singular solution". On the other hand, we have a two parameter family of solutions :

$$\gamma_{(c_1,c_2)}(t) = (x, y, p) = (\frac{2}{3}c_1e^{-t} + c_2e^{2t}, \frac{1}{3}c_1^2e^{-2t} + 2c_1c_2e^t, c_1e^{-t}).$$

If we fix $p(0) = c_1 \neq 0$ and put $c = c_2c_1^2$, then we have $x = \frac{2}{3}p + \frac{c}{p^2}$ and $y = 2p \cdot x - p^2$. But, if we fix $p(0) = c_1 = 0$, then we have $y = 0$. Moreover, if we consider this family of

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