K. SAITO
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ALGEBRAIC REPRESENTATIONS OF TEICHMÜLLER SPACE

BY KYOJI SAITO

This is a summary of the paper [S5] with the same title. It is divided in two parts. In part I, we summarize some general results on the character variety of representations of a finitely generated group in SL_2 ([S4]). In part II, we apply these results to the representations of surface groups in $SL_2(\mathbf{R})$ to obtain a description of the coordinate ring for Teichmüller space as a semialgebraic set defined over $\mathbf{Z}$. Except for some minor changes, the introduction and §6 are the same as that of [ibd].

For integers $g \geq 0$ and $n \geq 0$, consider $(X, \underline{x})$, where X is a compact Riemann surface of genus g and $(\underline{x}) = (x_1, \ldots, x_n)$ is an n-tuple of distinct ordered points of X. Recall ([Mu2]) that the moduli space $\mathfrak{M}_{g,n}$ is the set of isomorphic classes of $(X, \underline{x})$, and that the Teichmüller space $\mathcal{T}_{g,n}$ is the set of isomorphic classes of $(X, \alpha, \underline{x})$, where α is an isomorphism from the abstract group $\Gamma_{g,n}$ with a fixed generator system (6.2.1) to $\pi_1(X \backslash \{\underline{x}\})$ satisfying certain marking properties (recalled in §6).

We recall some well known facts about the spaces $\mathfrak{M}_{g,n}$ and $\mathcal{T}_{g,n}$, which led the author to the present work. $\mathfrak{M}_{g,n}$ has the structure of the $\mathbf{C}$-rational point set of a quasi-projective algebraic variety defined over $\mathbf{Q}$ ([Mu]) and $\mathcal{T}_{g,n}$ has the structure of a smooth real manifold so that the natural projection $\mathcal{T}_{g,n} \to \mathfrak{M}_{g,n}$ is a ramified covering map, having $\mathrm{Out}^+(\Gamma_{g,n})$ (6.2.3) as the covering transformation group. $\mathcal{T}_{g,n}$ is homeomorphic to a cell, and hence it is the universal covering space of $\mathfrak{M}_{g,n}$ in the sense of orbifold. The covering map cannot be algebraic (in any sense) when the group $\mathrm{Out}^+(\Gamma_{g,n})$ is infinite, in particular when $e(X \backslash \{\underline{x}\}) = 2 - 2g - n < 0$. Therefore, it is interesting to ask for a construction of an automorphic function (or an automorphic form in a suitable sense) on $\mathcal{T}_{g,n}$ with respect to the action of $\mathrm{Out}^+(\Gamma_{g,n})$. After such an attempt in [S2], the author was led to fix certain real algebraic coordinates of $\mathcal{T}_{g,n}$ which fits with that construction. This motivated the present paper.

In fact, we will realize $\mathcal{T}_{g,n}$ for $2 - 2g - n < 0$ as the $\mathbf{R}$-rational point set of an affine semialgebraic set defined over $\mathbf{Z}$ ((6.5) Theorem), where the coordinate ring $R_{g,n}$ is introduced abstractly as a quotient ring of *the universal character ring* $R(\Gamma_{g,n}, SL_2)$ (introduced in part I) modulo the ideal corresponding to the parabolicity conditions on the n generators of $\Gamma_{g,n}$ (6.5.1). The image of the ring $R_{g,n}$ in $\mathbf{R}$ at a point $(X, \alpha, \underline{x})$ of $\mathcal{T}_{g,n} \subset \mathrm{Hom}(R_{g,n}, \mathbf{R})$ is an invariant of the n-pointed Riemann surface $(X, \underline{x})$, which we call the *ring of uniformization* ((6.3.1)).

Historically,the real algebraic description of Teichmüller space was initiated by Fricke [F-K], who used characters of discrete subgroups in $SL_2(\mathbf{R})$ in order to determine the moduli space of compact Riemann surfaces. So some authors call it the Fricke moduli.

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