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ON THE SMOOTH STRUCTURES ON ELLIPTIC SURFACES AND RELATED TOPICS

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A compact complex surface is called an elliptic surface if it admits a holomorphic projection to a Riemann surface whose general fiber (the inverse image of a regular value of the projection) is a nonsingular elliptic curve. Some of the total spaces of the elliptic surfaces as closed topological 4-manifolds have infinitely many smooth structures. Therefore they provide typical examples of exotic phenomena which occur only in dimension 4. In particular to treat the homeomorphism types and the diffeomorphism types of the elliptic surfaces when their fundamental groups are finite cyclic we need the deep results originated from Freedman and Donaldson. Whereas in case of those whose fundamental groups are not finite cyclic, they can be treated in more elementary way although in general we have no theories applicable to all 4-manifolds with given fundamental groups. Moreover there are still very few 4-manifolds other than the elliptic surfaces whose Donaldson invariants are explicitely computed (although they are not completely known even in the elliptic surfaces). So in this note we will describe some of such topics. In §1 we collect some known facts about the homeomorphism types and the diffeomorphism types of the elliptic surfaces. In §2 we describe the smooth structures on some elliptic surfaces not coming from complex surfaces and "exotic" free actions on some 4-manifolds derived from them ([U3], [U4]). The details of their proofs will appear elsewhere.

§1. Some known facts about elliptic surfaces

Let $\pi : S \to B$ be a compact elliptic surface over an Riemann surface B. Here we can assume without loss of generality that S is relatively minimal. This means that S has no (-1)-curve in any fiber. This is because if S is not relatively minimal then the finitely many blowing downs provide a relatively minimal one. So we assume this for the rest of this note. Furthermore we will concentrate on the topology of the total space S of the elliptic surface and so we ignore its complex structure itself (in particular any diffeomorphic elliptic surfaces are identified in this note). For more comprehensive investigation about the topology of them including their deformation types, see the book of Friedman and Morgan ([FM4]).

Let $e(S)$ be the euler number of S. Then it is known (by Noether formula) that $e(S) \geqq 0$ and it must be a multiple of 12. Then S belongs to one of the following classes.

(I) $e(S) = 0$

An elliptic surface belongs to this class if and only if it contains only multiple tori (defined below) as its singular fibers. The elliptic surfaces in this case were completely

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