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QUASISYMMETRIC MAPS AND STRING THEORY

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Abstract

We survey some recent research on the geometry of Lipman Bers' universal Teichmüller space $T(1)$, i.e., the space of quasisymmetric homeomorphisms of the circle modulo Möbius moves, and its applications in string theory.

1. Geometric quantization

Bosonic string theory [9, 13, 26] is a proposal of unified field theory where the elementary particles called bosons are supposed to appear as 1-dimensional extended objects in the Planck scale 10^{-35} m; hence, topologically they look like either $\mathbb{R}$ (open string) or S^1 (closed string). We shall work with closed strings. The string hypothesis introduces a new symmetry group into physics, the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of the circle, as this is the internal symmetry group of a closed string. *Non-perturbative bosonic string theory* would be based, ideally at least, on the group $\mathrm{Homeo}(S^1)$. We would like to geometrize this group, but as it seems to be intractable, in practice, we need to content ourselves with some subgroup.

There is a standard procedure in physics called *geometric quantization* [36] to pass from a classical system to a quantum system. In the classical system, the ***observables*** are functions f in the ***phase space*** which is a smooth manifold M^{2n} endowed with a symplectic form ω; in the corresponding quantum system the observables need to be converted into operators T_f acting in some Hilbert space in such a way that Poisson brackets of functions are converted into Lie brackets of operators

$$T_{\{f_1,f_2\}} = [T_{f_1}, T_{f_2}]. \tag{1.1}$$

The standard way to achieve this is to produce a Hermitian line bundle $\mathcal{L}$ over M with a Hermitian connection ∇ whose curvature equals ω. Then the sought-for operators will be given by

$$T_f = -i\nabla_{X_f} + f \tag{1.2}$$

where X_f is the Hamiltonian vector field corresponding to the observable f by the formula $X_f = -\omega^{-1}(df, .)$. The operators T_f act in the Hilbert space of square-integrable sections of $\mathcal{L}$ with respect to the canonical volume form $\frac{\omega^n}{n!}$ of (M, ω). In fact, up to this point, we have only achieved *prequantization* while the difficult *Dirac problem* concerning the irreducibility of the representation $f \mapsto T_f$ remains to be settled. This final step in the geometric quantization programme can often be achieved by introducing a

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