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NOTE ON ESTIMATION OF THE NUMBER OF THE CRITICAL VALUES AT INFINITY

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1. Let $f(x,y)$ be a polynomial of degree d and we consider the polynomial function $f:\mathbf{C}^2 \to \mathbf{C}$. Let $\Sigma(f)$ be the critical values. The restriction

$$f:\mathbf{C}^2 - f^{-1}(\Sigma) \to \mathbf{C} - \Sigma$$

is not necessarily a locally trivial fibration. We say that $\tau \in \mathbf{C}$ is *a regular value at infinity* of the function $f:\mathbf{C}^2 \to \mathbf{C}$ if there exist positive numbers R and ε so that the restriction of f, $f: f^{-1}(D_\varepsilon(\tau)) - B_R^4 \to D_\varepsilon(\tau)$, is a trivial fibration over the disc $D_\varepsilon(\tau)$ where $D_\varepsilon(\tau) = \{\eta \in \mathbf{C}; |\eta - \tau| \le \varepsilon\}$ and $B_R^4 = \{(x,y); |x|^2 + |y|^2 \le R\}$. Otherwise τ is a called *a critical value at infinity.* We denote the set of the critical values at infinity by Σ_∞. It is known that Σ_∞ is finite ([23], [2]). The purpose of this note is to give an estimation on the number of critical values at infinity. The detail will be published elsewhere ([12]).

We first consider the canonical projective compactification $\mathbf{C}^2 \subset \mathbf{P}^2$. We denote the homogeneous coordinates of $\mathbf{P}^2$ by X, Y, Z so that $x = X/Z$ and $y = Y/Z$ Let L_∞ be the line at infinity: $L_\infty = \{Z = 0\}$. Write

$$f(x,y) = f_0 + f_1(x,y) + \cdots + f_d(x,y)$$

where $f_i(x,y)$ is a homogeneous polynomial of degree i for $i = 0,\dots,d$. We can write

$$f_d(x,y) = cx^{\nu_0}y^{\nu_{k+1}}\prod_{j=1}^{k}(y - \lambda_j x)^{\nu_j} \tag{1.1}$$

where $c \in \mathbf{C}^*$ and $\lambda_1,\dots,\lambda_k$ are non-zero distinct numbers and we assume that $\nu_i > 0$ for $1 \le i \le k$ and $\nu_0, \nu_{k+1} \ge 0$. Note that we have the equality

$$\nu_0 + \cdots + \nu_{k+1} = d \tag{1.2}$$

Let C_τ be the projective curve which is the closure of the fiber $f^{-1}(\tau)$. Then C_τ is defined by $C_\tau = \{(X;Y;Z) \in \mathbf{P}^2; F(X,Y,Z) - \tau Z^d = 0\}$ where $F(X,Y,Z)$ is the homogeneous polynomial defined by

$$F(X,Y,Z) = f(X/Z, Y/Z)Z^d = f_0 Z^d + f_1(X,Y)Z^{d-1} + \cdots + f_d(X,Y) \tag{1.3}$$

The intersection of C_τ and the line at infinity, $C_\tau \cap L_\infty$, is independent of $\tau \in \mathbf{C}^2$ and it is the base point locus of the family $\{C_\tau; \tau \in \mathbf{C}\}$. Obviously we have $C_\tau \cap L_\infty = \{Z = f_d(X,Y) = 0\}$. For brevity, let $A_i = (\alpha_i;\beta_i;0) \in \mathbf{P}^2$ for $i = 0,\dots,k+1$ where $A_0 = (0;1;0), A_{k+1} = (1;0;0)$ and $\beta_i/\alpha_i = \lambda_i$ for $1 \le i \le k$. Then under the assumption (1.1), $C_0 \cap L_\infty = \{A_i; \nu_i > 0\}$. Note that $A_i \in C_0 \cap L_\infty$ for $i = 1,\dots,k$. We consider the family of germs of a curve at A_j: $\{(C_\tau, A_j); \tau \in \mathbf{C}\}$. Then it is known that τ is a

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