NGÔ VIÊT TRUNG
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# AN ALGEBRAIC APPROACH TO THE REGULARITY INDEX OF FAT POINTS IN $\mathbf{P}^n$

By Ngô Viêt Trung

**Abstract**

The aim of this note is to present an efficient algebraic method for the estimation of the (Castelnuovo) regularity index of fat points in $\mathbf{P}^n$ and to discuss recent results on this topic.

## Introduction

Given a point $P$ in the projective space $\mathbf{P}^n := \mathbf{P}^n(k)$, $k$ an algebraically closed field, we say that a form (or a hypersurface) $f$ of the polynomial ring $R := k[X_0, \dots, X_n]$ has multiplicity $m$ at $P$ if all derivatives of $f$ of order $\leqq m$ vanish at $P$.

Let $X = \{P_1, \dots, P_s\}$ be a set of $s$ points in $\mathbf{P}^n$ and $m_1 \geqq \dots \geqq m_s$ a sequence of positive integers. We denote by $m_1P_1 + \cdots m_sP_s$ the zero-scheme defined by the ideal of all forms of $R$ vanishing at $P_i$ with multiplicity $\geqq m_i$, $i = 1, \dots, s$, and call

$$Z := m_1P_1 + \cdots m_sP_s.$$

a set of *fat points* in $\mathbf{P}^n$. There are many reasons for our interest in this notion. For instance, $Z$ is simply the zero-scheme of hypersurfaces passing through $X$ if $m_1 = \dots = m_s = 1$, and of hypersurfaces containing $P_1, \dots, P_s$ as singular points if $m_1 = \dots = m_s = 2$.

Let $\mathcal{I}$ denote the ideal sheaf of $Z$. Then $H^0(\mathbf{P}^n, \mathcal{I}(t))$ corresponds to the linear system of hypersurfaces of degree $t$ passing through $P_1, \dots, P_s$ with multiplicity $\geqq m_i$ at $P_i$, $i = 1, \dots, s$. If $H^1(\mathbf{P}^n, \mathcal{I}(t)) = 0$, this linear system is called regular. The least integer $t$ for which $H^1(\mathbf{P}^n, \mathcal{I}(t)) = 0$ is called the *regularity index* of $Z$, and we will denote it by $r(Z)$. There has been much interest to estimate $r(Z)$ in terms of $m_1, \dots, m_s$.

For arbitrary fat points in $\mathbf{P}^2$ one can find in W. Fulton [6] the following upper bound:

$$r(Z) \leqq \sum_{i=1}^{s} m_i - 1.$$

This bound was extended to arbitrary fat points in $\mathbf{P}^n$ by E. Davis and A. Geramita in [4], where they also showed that $r(Z) = \sum_{i=1}^{s} m_i - 1$ if and only if $P_1, \dots, P_s$ lie on a line of $\mathbf{P}^n$. For almost all sets $X$ of $s$ points in $\mathbf{P}^2$, B. Segre [16] found the upper bound:

$$r(Z) \leqq \max\{m_1 + m_2 - 1, [\sum_{i=1}^{s} m_i/2]\}.$$

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