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HARMONIC DIMENSION AND EXTREMAL LENGTH

Dedicated to Professor Nobuyuki Suita on his sixtieth birthday

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Consider an open Riemann surface R with a single ideal boundary component. A subregion $V(\neq R)$ of R is said to be an *end* of R if V is relatively noncompact in R and the relative boundary ∂V consists of finitely many analytic Jordan curves. Denote by $\mathcal{P}(V)$ the class of nonnegative harmonic functions on V with vanishing boundary values on ∂V:

$$\mathcal{P}(V)=\{h \in HP(V): h \mid \partial V=0\},$$

where $HP(V)$ is the class of nonnegative harmonic functions on V. The dimension of the linear space $\mathcal{P}(V) \ominus \mathcal{P}(V)=\{h_1-h_2: h_1, h_2 \in \mathcal{P}(V)\}$ is referred to as the *harmonic dimension* of V (cf. Heins [4]), $\dim \mathcal{P}(V)$ in notation. It is known that $\dim \mathcal{P}(V)$ does not depend on a choice of an end V of R (cf. [4]): $\dim \mathcal{P}(V_1)$ $=\dim \mathcal{P}(V_2)$ for any pair (V_1, V_2) of ends of R.

Denote by O_G the class of open Riemann surfaces of null boundary and by M the class of open Riemann surfaces $R \in O_G$ such that there exists an end V of R with $\dim \mathcal{P}(V)=1$. In terms of Martin compactification an R belongs to M if and only if R is of null boundary and the Martin boundary of R consists of a single point (cf. e.g. Constantinescu and Cornea [3]). We are particularly interested in the following result by Heins [4] (see also [7]):

THEOREM A. *Let V be an end and $\{A_n\}$ be a sequence of mutually disjoint annuli in V satisfying that A_{n+1} separates A_n from the ideal boundary of V for every n. If the sum of moduli of A_n diverges, then* $\dim \mathcal{P}(V)=1$.

We also denote by $O_G^{\prime\prime}$ the class of open Riemann surfaces having a regular exhaustion $\{R_n\}_{n=0}^{\infty}$ such that each $A_n=R_{2n}-\overline{R_{2n-1}}$ $(n=1, 2, \cdots)$ is a doubly connected region and $\sum_{n=1}^{\infty} \operatorname{mod} A_n=\infty$, where $\operatorname{mod} A_n$ is the modulus of A_n. Then the above Heins' result is restated as

$$O_G^{\prime\prime} \subset M .$$

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