F. ICHIKAWA
KODAI MATH. J.
15 (1992), 134—140

THOM'S CONJECTURE ON SINGULARITIES OF GRADIENT VECTOR FIELDS

BY FUMIO ICHIKAWA

1. Introduction.

In [3], R. Thom gave the following conjecture.

CONJECTURE. *Let $f(x)$ be a germ of real analytic function at the origin $0\in R^n$ and let $X=\operatorname{grad} f(x)$ be the gradient vector field of $f(x)$ with respect to the ordinary Riemannian metric on R^n. If an integral curve $g(t)$ of X tends to the origin $0\in R^n$, then there exists a unique tangential direction* $\lim_{t\to+\infty} g(t)/|g(t)|$.

Thom proved the case where $f(x)$ is a homogeneous polynomial and for the general case he gave an outline of a proof. In this paper, we give a partial answer to the above problem. The essential idea of our proof is the same as Thom's one (see [3]).

Let $f(x):(R^n, 0)\to(R, 0)$ be a germ of analytic function. And we express $f(x)$ in the form

$$f(x)=P_k(x)+P_{k+1}(x)+\cdots+P_m(x)+\cdots$$

where $P_m(x)$ is a homogeneous polynomial of degree m.

We define the *cone spectrum* $Sp(P_m)$ as follows:

$$Sp(P_m)=\left\{x=(x_1, \cdots, x_n)\in R^n;\ x_i\frac{\partial P_m}{\partial x_j}=x_j\frac{\partial P_m}{\partial x_i}\quad i, j=1, \cdots, n\right\}.$$

Obviously, $Sp(P_m)$ is a cone algebraic set and it contains $0\in R^n$.

In this paper, we prove the following theorem.

THEOREM. *Let $f(x)=P_k(x)+P_{k+1}(x)+\cdots$ be a real analytic function germ at $0\in R^n$. If* $\dim Sp(P_k)\leqq 1$, *then any integral curve of* $\operatorname{grad} f(x)$ *which tends to $0\in R^n$ has a unique tangential direction at the origin.*

Remark. We see later on that the condition $\dim Sp(P_k)\leqq 1$ is equivalent to that the restricted function $P_k|_{S^{n-1}}$ of $P_k(x)$ to the unit sphere S^{n-1} has only isolated singularities. Thus, the above condition is a generic property on the initial term of $f(x)$.

Received June 21, 1991.