T. WATANABE
KODAI MATH. J.
15 (1992), 82—101

# SUFFICIENT CONDITIONS FOR UNIMODALITY OF NON-SYMMETRIC LÉVY PROCESSES

BY TOSHIRO WATANABE

## 1. Introduction and results.

A measure $\mu$ on $R^1$ is said to be *unimodal* with mode $a$ if $\mu(dx)=c\delta_a(dx)+f(x)dx$, where $c\geqq 0$, $\delta_a$ is the delta measure at $a$, $f(x)$ is non-decreasing for $x<a$ and non-increasing for $x>a$. A probability measure $\mu$ on $R^1$ is said to be *strongly unimodal* if, for every unimodal probability measure $\eta$, the convolution $\mu*\eta$ is unimodal. Let $\{X_t\}$ $(t\geqq 0)$ be a Lévy process on $R^1$ (that is a process with stationary independent increments starting at the origin). The process $\{X_t\}$ is said to be of class $L$ if the distribution $\mu_t$ of $X_t$ is of class $L$ for every $t>0$ (equivalently, for some $t>0$). A necessary and sufficient condition for an infinitely divisible distribution $\mu$ with Lévy measure $\nu$ to be of class $L$ is that $|x|\nu(dx)$ is unimodal with mode 0. The process $\{X_t\}$ is said to be unimodal if the distribution $\mu_t$ is unimodal for every $t>0$. Medgyessy [1] and Wolfe [13] show that symmetric Lévy processes are unimodal if and only if their Lévy measures are unimodal with mode 0. Yamazato [14] proves that every process of class $L$ is unimodal. Watanabe [8] shows that there exist unimodal non-symmetric Lévy processes that are not of class $L$. Also, Watanabe [10] gives a necessary and sufficient condition for unimodality of one-sided Lévy processes by using zeros of some polynomials. However it has not been successful to find a necessary and sufficient condition in terms of their Lévy measures. Other results on the unimodality of Lévy processes are obtained by Sato [2, 3], Sato-Yamazato [4], Steutel-van Harn [6], Watanabe [9, 11], Wolfe [12], and Yamazato [15]. The purpose of this paper is to improve the previous paper [8] and to give sufficient conditions for unimodality of non-symmetric Lévy processes that are not of class $L$, in terms of their Lévy measures. To describe our results, we need to introduce some notations.

From now on, let $n$ be a positive integer,

$$0=b_0<a_1<b_1<a_2<b_2<\cdots<a_n<b_n<a_{n+1}\leqq\infty,$$

and let $k(x)$ be a function on $(0, \infty)$ such that $k(0+)<\infty$, $k(x)>0$ on $(0, a_{n+1})$, $k(x)=0$ on $[a_{n+1}, \infty)$, $k(x)$ is non-increasing on $[b_m, a_{m+1}]$ $(0\leqq m\leqq n)$, non-

Received May 13, 1991.