S. DENG
KODAI MATH. J.
14 (1991), 470—476

THE SECOND VARIATION OF THE DIRICHLET ENERGY ON CONTACT MANIFOLDS

BY SHANGRONG DENG

1. Introduction

S. S. Chern and R. S. Hamilton in a paper of 1985 [5] studied a kind of Dirichlet energy in terms of the torsion $\tau(\tau=\mathcal{L}_\xi g)$ of a 3-dimensional compact contact manifold and a problem analogous to the Yamabe problem. They raised the question of determining all 3-dimensional contact manifolds with $\tau=0$ (i.e. K-contact). In a long paper of 1989 [8] S. Tanno studied the Dirichlet energy and gauge transformations of contact manifolds. D. E. Blair [2] obtained the critical point condition of $I(g)=\int_M Ric(\xi)dV_g$ over $\mathcal{M}(\eta)$ (the space of all the associated metrics), and proved that the regularity of the characteristic vector field ξ and the critical point condition force the metric to be K-contact. Since $Ric(\xi)=2n-1/4|\tau|^2$, the study of $I(g)$ is the same as the study of the Dirichlet energy. In this paper we investigate the second variation and prove the following result.

THEOREM 2. *Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy $L(g)=\int_M |\tau|^2 dV_g$, i.e. $\nabla_\xi L_\xi g=2(\mathcal{L}_\xi g)\phi$, then along any path $g_{ij}(t)=g_{ir}[\delta^r_j+tH^r_j+t^2K^r_j+O(t^3)]$ in $\mathcal{M}(\eta)$*

$$\frac{d^2L}{dt^2}(0)=2\int_M |\mathcal{L}_\xi H^i_j|^2 dV_g \geqq 0\,,$$

and $L(g)$ has minimum at each critical metric.

The author would like to thank Professor David E. Blair for his constant encouragement and help.

2. Contact manifolds

A C^∞ manifold M^{2n+1} is said to be a *contact manifold* if it carries a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere. Given a contact form η it is well

Received February 5, 1991; revised April 15, 1991.