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ON SLOWLY INCREASING UNBOUNDED HARMONIC FUNCTIONS

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Abstract

In this paper we prove several growth results for slowly increasing unbounded harmonic functions in the unit disc. They generalize some of the theorems in [4] for our new definition of maximum growth in a finite number of directions.

§1. Introduction.

Let $u(z)$ be a harmonic function in a plane domain Ω, consider the level curves of $u(z)$, $l(c)=\{z\in\Omega: u(z)=c\}$ for $-\infty<c<\infty$, and let $\Theta(c)=\int_{l(c)}|*du|$ for $-\infty<c<\infty$, where $*du$ is the differential of the harmonic conjugate function of $u(z)$, with the agreement that $\Theta(c)=0$ if $l(c)=\emptyset$. We have the following definition.

DEFINITION A. Let $u(z)$ be a harmonic function in a domain Ω, if there exists $a\in u(\Omega)$ such that, $\int_a^b \frac{dc}{\Theta(c)}<\infty$ for every $b>a$, and $\lim_{b\to\infty}\int_a^b \frac{dc}{\Theta(c)}=\infty$, then we say that the function $u(z)$ is a slowly increasing unbounded harmonic function.

Observe that by our convention for any bounded harmonic function the above integral will take the value infinity for a finite value of b. If the function grows to infinity fast as we approach the boundary of Ω the integral $\int_a^b \frac{dc}{\Theta(c)}$ increases more slowly, thus in a sense the functions defined above are the most slowly increasing unbounded harmonic functions.

Consider the family of curves $\Gamma(a, b)=\{l(c): a<c<b\}$ for $-\infty\leqq a\leqq b<\infty$. We denote its module by the symbol $\mu(a, b)$, then it is known that if $\Gamma(a, b)\neq\emptyset$ and $(a, b)\subset u(\Omega)$ then

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