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A CLASSIFICATION OF 3-DIMENSIONAL CONTACT METRIC MANIFOLDS WITH $Q\varphi=\varphi Q$

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1. Introduction

The assumption that $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a contact metric manifold is very weak, since the set of metrics associated to the contact form η is huge. Even if the structure is η-Einstein we do not have a complete classification. Also for $m=1$, we know very little about the geometry of these manifolds [8]. On the other hand if the structure is Sasakian, the Ricci operator Q commutes with φ ([1], p. 76), but in general $Q\varphi\neq\varphi Q$ and the problem of the characterization of contact metric manifolds with $Q\varphi=\varphi Q$ is open. In [13] Tanno defined a special family of contact metric manifolds by the requirement that ξ belong to the k-nullity distribution of g. We also know very little about these manifolds (see [13] and [9]). In § 3 of this paper we first prove that on a 3-dimensional contact metric manifold the conditions, i) the structure is η-Einstein, ii) $Q\varphi=\varphi Q$ and iii) ξ belongs to the k-nullity distribution of g are equivalent. We then show that a 3-dimensional contact metric manifold on which $Q\varphi=\varphi Q$ is either Sasakian, flat or of constant ξ-sectional curvature k and constant φ-sectional curvature $-k$. Finally we give some auxiliary results on locally φ-symmetric contact metric 3-manifolds and on contact metric 3-manifolds immersed in a 4-dimensional manifold of contant curvature $+1$.

2. Preliminaries

A C^∞ manifold M^{2m+1} is said to be a *contact manifold*, if it carries a global 1-form η such that $\eta\wedge(d\eta)^m\neq 0$ everywhere. We assume throughout that all manifolds are connected. Given a contact form η, it is well known that there exists a unique vector field ξ, called the *characteristic vector field* of η, satisfying $\eta(\xi)=1$ and $d\eta(\xi, X)=0$ for all vector fields X. A Riemannian metric g is said to be an *associated metric* if there exists a tensor field φ of type (1, 1) such that

$$d\eta(X, Y)=g(X, \varphi Y),\ \eta(X)=g(X, \xi),\ \varphi^2=-I+\eta\otimes\xi. \tag{2.1}$$

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