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ON THE LIE CURVATURE OF DUPIN HYPERSURFACES

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0. Introduction.

A hypersurface M in a standard sphere S^n is said to be *Dupin* if each of its principal curvatures is constant along its corresponding curvature surfaces. If the number of distinct principal curvatures is constant, then M is called a *proper* Dupin hypersurface. There is a close relationship between the class of compact proper Dupin hypersurfaces and the class of isoparametric hypersurfaces. Münzner [11] showed that the number g of distinct principal curvatures of an isoparametric hypersurface must be 1, 2, 3, 4 or 6. Thorbergsson [15] then showed that the same restriction holds for a compact proper Dupin hypersurface embedded in S^n by reducing that case to a situation where Münzner's argument can be applied. This also implied that the rank of the Z_2-cohomology ring in both cases must be $2g$. Later Grove and Halperin [6] found more topological similarities between these two classes of hypersurfaces. All of this led to the conjecture [5, p. 184] that every compact proper Dupin hypersurface in S^n is equivalent by a Lie sphere transformation to an isoparametric hypersurface.

The conjecture was known to be true in the cases $g=1$ (umbilic hypersurfaces), $g=2$[4] and $g=3$[7]. Recently, however, counterexamples to the conjecture for $g=4$ have been discovered by Miyaoka and Ozawa [10] and by Pinkall and Thorbergsson [14]. Miyaoka and Ozawa also produced counterexamples in the case $g=6$. In all cases, the proof that the counterexamples are not Lie equivalent to an isoparametric hypersurface uses the so-called Lie curvature Ψ introduced by Miyaoka [8]. For a proper Dupin hypersurface M with four principal curvatures, Ψ is the cross-ratio of these principal curvatures. Viewed in the context of Lie sphere geometry, Ψ is the cross-ratio of the four points on a projective line corresponding to the four curvature spheres of M. Hence, Ψ is a natural Lie invariant. From Münzner's work, it is easy to compute that Ψ has a constant value 1/2 on a Dupin hypersurface which is Lie equivalent to an isoparametric hypersurface. In projective geometric terms, this means that the four curvature spheres at each point of M form a harmonic set. For the counterexamples to the conjecture above, Ψ does not have the constant value 1/2.

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