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ON THE STABILITY OF A THREE-SPHERE

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1. Introduction.

Let (M^m, g) be an m-dimensional closed connected Riemannian manifold. The identity mapping id_M of M is a harmonic mapping, that is, a critical point of the first variation of the energy functional. (M^m, g) is said to be stable when the second variation of the energy functional at id_M is non-negative and otherwise, (M, g) is said to be unstable. The m-dimensional $(m \geqq 3)$ unit spheres are unstable. And unstable, simply connected compact irreducible symmetric spaces were determined (see Smith [6], Nagano [4], Ohnita [5] and Urakawa [11]).

Closed manifolds with negative Ricci curvature and closed Kaehler manifolds are examples of stable manifolds. Since Gao and Yau [1] proved the existence of a metric with negative Ricci curvature on every 3-dimensional closed manifold, there exists a stable metric on every 3-dimensional closed manifold.

Recently Urakawa [12] and Tanno [9] studied some deformation of the standard metric g_0 on S^{2n+1} $(n \geqq 1)$ with constant sectional curvature one. Let (CP^n, h) be the complex projective space with the Fubini-Study metric with constant holomorphic sectional curvature 4 and $\pi : (S^m, g_0) \to (CP^n, h)$ $(m=2n+1)$ be the Hopf fibration. Let ξ be the unit Killing vector field on S^m which is tangent to each fibre and η be the dual 1-form of ξ with respect to g_0. We define a one-parameter family $g(t)$, $0<t<\infty$, of Riemannian metrics on S^m by

$$g(t)=t^{-1}g_0+t^{-1}(t^m-1)\eta\otimes\eta .$$

THEOREM (Tanno [9]) *For $m=2n+1 \geqq 3$ and $t>t_0(m)$, $(S^m, g(t))$ is unstable, where $t_0(m)=[\{(m^2-4)^{1/2}-1\}/(m^2-5)]^{1/m}$.*

In this note, we show:

THEOREM A. *$(S^3, g(t))$ is stable if and only if $t \leqq t_0(3)=[\sqrt{5}-1/4]^{1/3}= 0.676\cdots$.*

Remark 1. The sectional curvature $K_\sigma(t)$ of $g(t)$ is positive for $0<t<(4/3)^{1/3}$ (see Tanno [9]). In fact, for $t<1$, $t^4 \leqq K_\sigma(t) \leqq t(4-3t^3)$ and for $t \geqq 1$, $t(4-3t^3) \leqq K_\sigma(t) \leqq t^4$.

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