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ON JORIS' THEOREM ON DIFFERENTIABILITY OF FUNCTIONS

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1. Introduction.

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a function. If $f^2, f^3 \in C^\infty$, does it follow that $f \in C^\infty$? The Inverse Function Theorem does not immediately give the answer. In 1982 H. Joris answered this problem affirmatively by showing the following theorem.

THEOREM 1 (H. Joris [J]). *Let $n_1, n_2, \cdots, n_m$ be positive integers with g. c. d. $\{n_1, n_2, \cdots, n_m\}=1$. If $f: \boldsymbol{R}^n \to \boldsymbol{R}$ is a function such that $f^{n_i} \in C^\infty$ for $i=1, 2, \cdots, m$, then $f \in C^\infty$.*

In the same paper H. Joris proposed the next problem.

PROBLEM. *Find the other families of smooth functions $\{\phi_i : \boldsymbol{R} \to \boldsymbol{R} \mid i=1, 2, \cdots, m\}$ having the following property: For any function $f: \boldsymbol{R}^n \to \boldsymbol{R}$, f is smooth if and only if $\phi_i \circ f$ is smooth for $i=1, 2, \cdots, m$.*

If we assume the continuity of f, then we need only consider the germs at $x=0$ since the study of differentiability is a local problem. In 1985 J. Duncan, S.G. Krantz and H.R. Parks gave a certain family $\{\phi_i\}$ for continuous f.

THEOREM 2 (J. Duncan, S.G. Krantz and H.R. Parks, [D] Theorem 2). *Let $\phi_i : \boldsymbol{R} \to \boldsymbol{R}$ be smooth functions such that $\phi_i(x) = x^{n_i} +$ "higher order terms" near $x=0$ for $i=1, 2, \cdots, m$ with g.c.d. $\{n_1, n_2, \cdots, n_m\}=1$. Then $\{\phi_i\}$ has the following property: For any continuous function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ with $f(0)=0$, f is smooth near $x=0$ if and only if $\phi_i \circ f$ is smooth near $x=0$ for $i=1, 2, \cdots, m$.*

In the present paper, we give a simple proof of Joris' Theorem (§ 2) and the necessary and sufficient condition for $\{\phi_i\}$ to have the property mentioned in Theorem 2 (§ 3 Theorem 3). In Appendix (§ 4), we discuss this condition further, especially for polynomials ϕ_i.

2. Simple proof of Joris' Theorem.

The essential part of our proof is the following algebraic lemma.

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