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GAUGE FIELDS AND QUATERNION STRUCTURE

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Dedicated to Professor M. Obata on his 60th birthday

1. The aim of this article is to discuss geometry of the moduli space of Yang-Mills connections over a 4-manifold with quaternion structure.

Let (M, h) be a compact, connected Riemannian 4-manifold with covariantly constant almost complex structures $\{I_1, I_2, I_3\}$ satisfying $I_1I_2=-I_2I_1=I_3$. We call such almost complex structures covariantly constant quaternion structure. Note that only complex flat two-tori and Ricci flat $K3$ surfaces are such spaces.

Each almost complex structure I_i given on the base space M defines a 2-form θ_i on M which is covariantly constant; $\theta_i(X, Y)=h(I_iX, Y)$, $i=1, 2, 3$. The manifold M carries the canonical orientation compatible with the quaternion structure. The base metric h together with this orientation gives the Hodge operator $*$; $\Lambda^2(M)\to\Lambda^2(M)$, which is involutive. So the bundle $\Lambda^2=\Lambda^2(M)$ splits into $\Lambda^2=\Lambda^+ + \Lambda^-$($\Lambda^+$ and Λ^- are subbundles of self-dual 2-forms and of anti-self-dual 2-forms, respectively). Then over the manifold M Λ^+ becomes trivial. We have indeed the decomposition;

$$\Lambda^+=\boldsymbol{R}\theta_1\oplus\boldsymbol{R}\theta_2\oplus\boldsymbol{R}\theta_3 \tag{1.1}$$

Let P be a smooth principal bundle over the manifold M with a compact simple Lie group G.

Fix a positive number $l>4$ in order that analysis on gauge fields works well and denote by $\mathcal{A}=\mathcal{A}_P$ the set of all L_l^2 connections on P. The set $\mathcal{A}$ is an affine space with model vector space $\Omega^1(\mathfrak{g}_P)_l$, the space of L_l^2 1-forms over M taking values in the adjoint bundle $\mathfrak{g}_P=P\times_{Ad}\mathfrak{g}$($\mathfrak{g}$ is the Lie algebra of G). Then $\mathcal{A}=A+\Omega^1(\mathfrak{g}_P)_l$ for some fixed smooth connection A. The subset $\mathcal{A}_{ir}$ in $\mathcal{A}$ consisting of irreducible connections is dense and open relative to the L_l^2-topology. A connection is said to be irreducible if the centralizer of its holonomy group in G reduces to the center Z_G of G.

The group $\mathcal{G}=\mathcal{G}_P$ of L_{l+1}^2 gauge transformations of P acts on $\mathcal{A}$ smoothly as $g(A)=g^{-1}dg+g^{-1}\cdot A\cdot g$. Remark that $\mathcal{G}/_{Z_G}$ acts freely on $\mathcal{A}_{ir}$ so that by the slice argument $\mathcal{A}_{ir}$ has a fibration over the orbit space $\mathcal{B}_{ir}=\mathcal{A}_{ir}/(\mathcal{G}/_{Z_G})$ with fibre $\mathcal{G}/_{Z_G}$.

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