S. KODANI
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AN ESTIMATE ON THE VOLUME OF METRIC BALLS

BY SHIGERU KODANI

1. Introduction.

Let M be a complete Riemannian manifold of dimension n. We denote by $i(M)$ the injectivity radius of M, by $B(p, r)$ the metric ball in M of radius $r \leqq i(M)$ centered at $p \in M$ and by $\mathrm{vol}(B(p, r))$ the volume of $B(p, r)$. Furthermore we denote by $\alpha(n)$ the volume of the round sphere S^n of sectional curvature 1. M. Berger and J. Kazdan [3] showed that if M is closed then the volume $\mathrm{vol}(M)$ of M satisfies

$$\mathrm{vol}(M) \geqq \alpha(n)(i(M)/\pi)^n, \tag{1}$$

where the equality holds if and only if M is a round sphere of constant sectional curvature $(\pi/i(M))^2$. Later, C. B. Croke [6] showed that if M is closed then for $r \in [0, i(M)]$,

$$\operatorname*{Ave}_{x \in M} \mathrm{vol}(B(x, r)) \geqq \alpha(n)(r/\pi)^n. \tag{2}$$

Here the equality holds if and only if $r=i(M)$ and M is a round sphere. Here $\operatorname*{Ave}_{x \in M} f(x)$, for any function f on M, means $\frac{1}{\mathrm{vol}(M)}\int_M f(x)dx$. But it is believed that for any point $p \in M$ and for $r \in [0, i(M)]$,

$$\mathrm{vol}(B(p, r)) \geqq \alpha(n)(r/\pi)^n. \tag{3}$$

Here the equality holds if and only if $r=i(M)$, $B(p, i(M))=M$ and M is a round sphere. As partial results on this problem, not sharp lower bounds are already known ([1], [2] for $n=2, 3$ and [4] for all n). And under some restriction on the metric form, a sharp one is obtained by C. B. Croke [5]. Especially C. B. Croke [4] showed the following remarkable inequality,

$$\mathrm{vol}(B(p, r)) \geqq \left[\frac{\pi\alpha(n-1)}{n\alpha(n)}\right]^n \alpha(n)\left[\frac{r}{\pi}\right]^n. \tag{4}$$

Here

$$\left[\frac{\pi\alpha(n-1)}{n\alpha(n)}\right]^n \approx \left[\frac{\pi}{2n}\right]^{n/2}, \qquad n \to \infty. \tag{5}$$

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