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CONVEX CURVES WHOSE POINTS ARE VERTICES OF BILLIARD TRIANGLES

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Abstract

We find out convex curves (other than ellipses) all points of which are vertices of periodic orbits of billiard balls with period three.

0. Introduction.

Let C be a plane convex curve and let D be the reigion inside C. Let a point P move over D with constant speed along a straight line until it hits C where it is reflected so that the angle of reflection with C is equal to the angle of incidence. The motion appears in the geometrical optics and the billiard systems (cf. [1], [4]). We say that C has *constant width* if each point of C has a double normal. There exist C^∞ convex curves with constant width other than circles (cf. [3]). From the viewpoint of the geometrical optics and the billiard problems, the double normal property implies that all points of C are vertices of periodic orbits of billiard balls with period two. Combined with a property of homofocal ellipses, the theorem of Poncelet proves that all points of any ellipse are vertices of *billiard n-gons* for all $n \geqq 3$, i.e., periodic orbits of billiard balls with period n ([2], p. 196). It would be natural to ask whether the converse of this phenomenon is true. In the present note we will see that it is not true if the existence of billiard triangles is assumed alone. Namely, we construct C^∞ convex curves C other than ellipses such that all points of C are vertices of billiard triangles. The example we will show has the following properties: (a) There is a subarc A of C such that all points of A are vertices of billiard equilateral triangles. (b) All billiard triangles of C are isosceles triangles.

In Section 1 we derive the differential equation which the example of convex curves must satisfy. In Section 2 we give a special solution of the differential equation. We prove in Section 3 that if all billiard triangles are equilateral, then the convex curve C is a circle.

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