S. SAITOH
KODAI MATH. J.
9 (1986), 50—57

ON THE CONVOLUTION OF L_2 FUNCTIONS

BY SABUROU SAITOH

1. Introduction.

For the convolution $F*G$ of $F\in L_p(-\infty, \infty)(p\geqq 1)$ and $G\in L_1(-\infty, \infty)$, we know the fundamental inequality

$$\|F*G\|_p \leqq \|F\|_p \|G\|_1. \tag{1.1}$$

See, for example, [8, p. 3]. Note that for F, $G\in L_2(-\infty, \infty)$, in general, $F*G \notin L_2(-\infty, \infty)$. In this paper, we will give an identification of a Hilbert space spanned by the convolutions $F*G$ and establish fundamental inequalities in the convolution. Note that when the space is $L_2(0, \infty)$, the results are very simple and quite different from the present case $L_2(-\infty, \infty)$. See [7].

2. The case of functions with compact supports.

We first consider the case of the convolution $F*G$ of $F\in L_2(a, b)$ and $G\in L_2(c, d)$. Without loss of generality we assume that $a+d\leqq b+c$. Of course, in the convolution we regard F and G as zero in the outsides of the intervals $[a, b]$ and $[c, d]$, respectively. We consider the integral transform, for $F\in L_2(a, b)$ and $z=x+iy\in \boldsymbol{C}$

$$f(z)=\frac{1}{2\pi}\int_a^b F(t)e^{-izt}dt. \tag{2.1}$$

As we see from the general theory [5, 6] of integral transforms, the images $f(z)$ form the Hilbert space $H_{(a,b)}$ admitting the reproducing kernel on $\boldsymbol{C}$

$$K_{(a,b)}(z, \bar{u})=\frac{1}{2\pi}\int_a^b e^{-izt}e^{i\bar{u}t}dt. \tag{2.2}$$

Since the family $\{e^{-izt}; z\in \boldsymbol{C}\}$ is complete in $L_2(a, b)$, we further have the isometrical identity

$$\|f\|^2_{H(a,b)}=\frac{1}{2\pi}\int_a^b |F(t)|^2 dt. \tag{2.3}$$

Hence, by using the Fourier transform for (2.1) in the framework of the L_2 space, we have

Received March 16, 1985