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HADAMARD'S VARIATIONAL FORMULA FOR THE SZEGÖ KERNEL

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§1. **A variational formula.** The present note is concerned with the Hadamard's (first) variational formula for the Szegö kernel associated with a strictly pseudo-convex domain in $\boldsymbol{C}^n$ with $n \geqq 3$. A similar formula for the Bergman kernel has been given in [7].

Let $\Omega^0 \subset \boldsymbol{C}^n$ with $n \geqq 1$ be a bounded domain with smooth boundary $\partial\Omega^0$. Every smoothly perturbed domain of Ω^0 can be parametrized by a small function $\rho \in C^\infty(\partial\Omega^0; \boldsymbol{R})$ in such a way that the boundary of that domain Ω^ρ is given by

$$\partial\Omega^\rho = \{\zeta + \rho(\zeta)\boldsymbol{\nu}(\zeta);\ \zeta \in \partial\Omega^0\}, \tag{1}$$

where $\boldsymbol{\nu}(\zeta) = \partial/\partial\nu_\zeta$ denotes the unit exterior normal vector to Ω^0 at $\zeta \in \partial\Omega^0$ identified with an element of $\boldsymbol{C}^n$.

Let $S^\rho(z, w)$ for $z, w \in \Omega^\rho$ denote the Szegö kernel associated with Ω^ρ, which is the reproducing kernel associated with the space $L_b^2H(\Omega^\rho)$ of holomorphic functions in Ω^ρ with L^2 boundary values equipped with the $L^2(\partial\Omega^\rho)$ scalar product. With $\delta\rho \in C^\infty(\partial\Omega^0; \boldsymbol{R})$ and $z, w \in \Omega^\rho$ fixed arbitrarily, we set

$$\delta S^\rho(z, w) = \frac{d}{d\varepsilon} S^{\rho+\varepsilon\delta\rho}(z, w)|_{\varepsilon=0}, \tag{2}$$

which is the Hadamard's first variation of $S^\rho(z, w)$ at ρ in the direction $\delta\rho$. Our purpose is to show that, for a certain class of domains Ω^0, the variation (2) at $\rho=0$ exists and is given by

$$\begin{aligned} -\delta S^0(z, w) = &\int_{\partial\Omega^0} \frac{\partial}{\partial\nu_\zeta}[S^0(z, \zeta)S^0(\zeta, w)] \cdot \delta\rho(\zeta)d\sigma^0(\zeta) \\ &+ \int_{\partial\Omega^0} S^0(z, \zeta)S^0(\zeta, w)H^0(\zeta)\delta\rho(\zeta)d\sigma^0(\zeta), \end{aligned} \tag{3}$$

where $d\sigma^0(\zeta)$ denotes the induced surface measure of $\partial\Omega^0 \subset \boldsymbol{C}^n$ at ζ, and $H^0(\zeta)$ stands for the mean curvature of $\partial\Omega^0$ at ζ multiplied by $2n-1$. A concrete statement of our result is given as follows:

THEOREM. *If $\Omega^0 \subset \boldsymbol{C}^n$ is strictly pseudo-convex with $n \geqq 3$, then the variation (2) at $\rho=0$ exists and is given by* (3).

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