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COVARIANCE OPERATORS AND VON NEUMANN'S THEORY OF MEASUREMENTS

Dedicated to Prof. Kentaro Murata on his sixtieth birthday

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1. Introduction.

The Gaussian channels are defined in the following; Let H_1, H_2 be a pair of real separable Hilbert spaces and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be Borel fields of H_1, H_2, respectively. For this, let $\nu(\cdot, \cdot)$ be a real valued function defined on $H_1 \times \mathfrak{B}_2$ such as

(1) for each $x \in H_1$, $\nu(x, \cdot) \equiv \nu_x$ is a Gaussian measure on $\mathfrak{B}_2$ with mean vector $m_x \in H_2$ and covariance operator ρ_x on H_2,

(2) for each $B \in \mathfrak{B}_2$, $\nu(\cdot, B)$ is a measurable function on H_1.

Then the triple $[H_1, \nu, H_2]$ is said to be a Gaussian channel. In this paper, we consider the Gaussian channels constructed by the covariance operator ρ_x which is constant or not constant with respect to x and obtain the average mutual information of the compound source.

In particular in the case of the covariance operator ρ_x which is not constant with respect to x, we can give von Neumann's theory of measurements as the models. In 1962, Nakamura-Umegaki proved that the statistical development $\rho_1 \rightarrow \rho_2$ by the measurements is nothing but the conditional expectation in the sense of Umegaki and developed the theory of noncommutative integration. In this paper, restricting the case of real separable Hilbert spaces, we tryto obtain the average mutual information of the statistical development by identifying density operators with covariance operators of probability measures on real separable Hilbert spaces. And in the last section we define the relative entropy among density operators and study the properties of them. We remark that the properties of our defined relative entropy are similar to the properties of von Neumann's relative entropy in some sense. Though we use almost known results relative to probability measures on Hilbert spaces, the obtained results are based on the essence of the theory of noncommutative integration with respect to the special von Neumann algebra $L(H)$.

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