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NORM INEQUALITIES OF EXPONENTIAL TYPE FOR HOLOMORPHIC FUNCTIONS

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Abstract. Let $\varDelta_\rho=\{z\in \boldsymbol{C}: |z|<\rho\}$ with $\rho=1, \infty$ and let $H(\varDelta_\rho)$ stand for the class of holomorphic functions in $\varDelta_\rho$. Let $\phi\in H(\varDelta_\rho)$ with $\varDelta_\rho$ being the domain of holomorphy of ϕ and $\phi(0)=0$, $\phi^{(n)}(0)>0$ for $n=1, 2, \cdots$. Then $k(z, \zeta)\equiv\phi(z\bar{\zeta})$ is the reproducing kernel of a uniquely determined Hilbert space H_ϕ of functions $f\in H(\varDelta_\rho)$ with $f(0)=0$ and norm $\|f\|_\phi$. The function $\psi\equiv\exp\phi$ also determines a unique Hilbert space H_ψ of functions $g\in H(\varDelta_\rho)$ with norm $\|g\|_\psi$ and such that $K(z, \zeta)\equiv\psi(z\bar{\zeta})$, $z, \zeta\in\varDelta_\rho$, is its reproducing kernel. The following is proved: Let $f\in H_\phi$, then $\exp f\in H_\psi$ and

$$\|\exp f\|_\psi^2\leqq\exp\|f\|_\phi^2$$

with equality if and only if f is of the form $f(z)=k(z, \zeta)=\phi(z\bar{\zeta})$ for some $\zeta\in\varDelta_\rho$. The method of proof of this sharp inequality is based on ideas of both Aronszajn and Milin, and it can be extended by replacing the exponential function by any entire function with non-negative Taylor-coefficients. We also give several applications of this inequality in the theory of entire functions and functions holomorphic in the unit disk.

1. Introduction.

Let $\varLambda$ be an abstract non-void set and let $k(\cdot, \cdot)$ be a scalar-valued kernel on $\varLambda\times\varLambda$. For simplicity, we always assume that the underlying scalar-field is the complex-field $\boldsymbol{C}$. We also assume that $k(\cdot, \cdot)$ is a positive-definite kernel on $\varLambda\times\varLambda$; that is

$$\sum_{m,n=1}^{N} k(z_m, z_n)\alpha_m\bar{\alpha}_n\geqq 0$$

for any finite set $\{z_m\}_{m=1}^N$ of points of $\varLambda$ and any corresponding complex numbers $\{\alpha_m\}_{m=1}^N$. As is well-known, this condition is equivalent to the existence of a uniquely determined Hilbert space H_k of functions on $\varLambda$ and admitting $k(\cdot, \cdot)$ as a reproducing kernel, namely

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