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RATIONAL APPROXIMATION AND SWISS CHEESES OF POSITIVE AREA

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Abstract

Let J and K be two compact sets in the complex plane such that $K \backslash J$ has zero planar measure. If $R(J)=C(J)$ then $R(K)=C(K)$. This result is used to produce many Swiss cheeses K of positive area, for which $R(K)=C(K)$.

For any compact set K in the complex plane, let $C(K)$ and $R(K)$ denote, respectively, the algebra of continuous functions on K, and the subalgebra of functions which are uniformly approximable on K by rational functions with poles off K. Hartogs and Rosenthal proved in [2] that if $m_2(K)=0$ (where m_2 denotes planar Lebesgue measure), then $R(K)=C(K)$. We extend this theorem here, and apply it to get new examples of Swiss cheeses K with $R(K)=C(K)$, yet $m_2(K)>0$.

THEOREM. *Let J and K be compact sets such that $m_2(K \backslash J)=0$. If $R(J)=C(J)$ then $R(K)=C(K)$.*

The proof of this result depends on the following. Let μ be a finite measure with compact support in the complex plane. The Cauchy transform of μ is defined by $\mu\hat{}(w)=\int(z-w)^{-1}d\mu(z)$. It is the convolution of μ with the locally integrable function $1/z$. So the integral defining $\mu\hat{}$ converges absolutely except for w belonging to a set of zero planar measure. Clearly, $\mu\hat{}$ is analytic off the closed support of μ. A converse of this statement is true.

PROPOSITION 1. (See [1], Theorem 8.2.) *Let μ be a finite measure of compact support in the plane. Suppose U is an open set, and f is a function analytic on U such that $f=\mu\hat{}$ almost everywhere with respect to m_2 on U. Then $|\mu|(U)=0$.*

Proof of Theorem 1. We show that any measure μ with support in K which is orthogonal to $R(K)$ must be the zero measure. In Proposition 1, set $f\equiv 0$ and $U=CJ$. Since $\mu \perp R(K)$, $\mu\hat{}=0$ on CK. Since $m_2(K \backslash J)=0$, we have $\mu\hat{}=f$ almost

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