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SYMMETRIC SPACES DERIVED FROM ALGEBRAS

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§ 1. Introduction.

The real projective plane is simply realized as the set of all lines through the origin in the 3-dimensional Euclidean space. And also it can be realized as the set of all subalgebras which are isomorphic to the field of complex numbers in the quaternion field. However there is a slight difference between two realizations, that is, in the former case the lines appear to have no algebraic structure but in the latter case the subalgebras do have it, by which the same symmetric space can be obtained. Then, since it seems to us that the similar realization to the latter is suitable for the explicit construction of symmetric spaces from various algebras, we will ask whether symmetric spaces in the sense of O. Loos [3] can be constructed by the set of all subalgebras with suitable conditions in a given algebra. In this paper we shall give an affirmative answer to this problem.

§ 2. Preliminaries.

Let K be the field of real numbers (or the field of complex numbers) and K^n be the n-dimensional vector space over K. We assume that K^n has a non-trivial product μ, i.e., a K-bilinear mapping $\mu: K^n \times K^n \to K^n$ such that $A \cdot A \neq \{0\}$ where we put $x \cdot y = \mu(x, y)$ and $A = (K^n, \mu)$. A is a non-associative algebra. Then the general linear group $GL(n, K)$ of K^n is a Lie group and the automorphism group $\mathrm{Aut}(A)$ of A, the group of all elements α of $GL(n, K)$ which satisfy $\mu(\alpha x, \alpha y) = \alpha\mu(x, y)$ for any $x, y \in A$, is also a Lie group because $\mathrm{Aut}(A)$ is a closed subgroup in $GL(n, K)$. Moreover we assume that A has a non-degenerate symmetric bilinear mapping (=inner product) $g: A \times A \to K$ which is invariant under $\mathrm{Aut}(A)$. Throughout this paper the product μ and the inner product g will be fixed.

A subspace V of the algebra A is regular if A is a direct sum of V and $V^\perp$ ($A = V \oplus V^\perp$) as a vector space where $V^\perp$ is a subspace of all elements x of A which are orthogonal to V relative to g, i.e., $g(x, V) = \{0\}$. Since the inner product g is symmetric and non-degenerate, for a regular subspace V, we can obtain a basis $\{e_i\}$ in A satisfying the conditions $(*)$: $e_i \in V$ $(1 \leq i \leq r)$, $e_i \in V^\perp$ $(r+1 \leq i \leq n)$

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