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ON A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION AND THE COSINE FUNCTION BY FACTORIZATION II

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1. In our previous paper [1] we proved the following

THEOREM A. *Let $F(z)$ be an entire function for which there exist polynomials $P_m(z)$ of degree m and entire functions $f_m(z)$ so that $F(z)=P_m(f_m(z))$ for $m=2^j$, $j=1, 2, \cdots$ and for $m=3$. Then $F(z)$ is either $Ae^{H(z)}+B$ or $A\cos\sqrt{H(z)}+B$ with constants A, B and an entire function $H(z)$.*

In this paper we shall give an application of this theorem.

THEOREM 1. *Let $F(z)$ be an entire function for which*

$$F(z)=P_2\left(F\left(\frac{z}{n}\right)\right)=P_3\left(F\left(\frac{z}{m}\right)\right)$$

with polynomials P_k of degree k and positive integers n, m. Then $F(z)$ is either $Ae^{az}+B$ or $A\cos az+B$ or $A\cos\sqrt{az}+B$ with constants A, B and a.

This theorem gives again a characterization of exp and cos. It seems to the present author that there is another proof depending on the power series expansion. If we omit the condition $F(z)=P_3\left(F\left(\frac{z}{m}\right)\right)$ in our theorem, we cannot say that theorem 1 holds.

In this theorem we may put m, n as non-zero constants and we have the same conclusion.

2. *Proof of Theorem* 1. Evidently

$$F(z)=P_{2^j}\left(F\left(\frac{z}{n^j}\right)\right)=P_3\left(F\left(\frac{z}{m}\right)\right)$$

for $j=1, 2, 3, \cdots$. Hence Theorem A implies that $F(z)$ is either $Ae^{H(z)}+B$ or $A\cos\sqrt{H(z)}+B$. By $F(z)=P_2(F(z/n))$ we have further $m(r, F)\sim 2m(r/n, F)$ as $r\to\infty$. For $r\geqq r_0$

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