M. OZAWA
KODAI MATH. J.
1 (1978), 45–74

ON A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION AND THE COSINE FUNCTION BY FACTORIZATION

BY MITSURU OZAWA

1. Introduction. Recently there have appeared quite many results in the theory of factorization by composition. However, so far as the present author knows, all of them are concerned with the possibility or the impossibility of factorization of certain functions. The factorization theory is still in the infancy so that almost all fundamental problems remain unsettled.

A meromorphic function $F(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be pseudo-prime if every factorization of the above form implies that $g(z)$ is a polynomial unless $f(z)$ is rational. If $F(z)$ is representable as $f_1(f_2 \cdots (f_n(z))\cdots)$ and $g_1(g_2 \cdots (g_n(z))\cdots)$ and if with suitable linear transformations λ_j, $j=1, \cdots, n-1$

$$f_1=g_1(\lambda_1),\ f_2=\lambda_1^{-1}(g_2(\lambda_2)),\ \cdots,\ f_n=\lambda_{n-1}^{-1}(g_n)$$

hold, then two factorizations are called to be equivalent.

It is well-known that e^z and $\cos z$ are both pseudo-prime and further they admit infinitely many non-equivalent left polynomial factors, that is, $e^z=w^n \circ e^{z/n}$ and $\cos z=P_n(\cos z/n)$ with a suitable polynomial P_n of degree n. This means that e^z and $\cos z$ occupy a quite special situation in the factorization theory. In this paper we shall discuss the inverse problem and prove the following characterization of the exponential function and the cosine function.

THEOREM. *Let $F(z)$ be an entire function, for which for every positive integer m there is a polynomial $P_m(z)$ of degree m such that*

$$F(z)=P_m(f_m(z))$$

for an entire function $f_m(z)$. Then

$$F(z)=A\cos\sqrt{H(z)}+B$$

with two constants A, B and an entire function $H(z)$, unless

Received March 24, 1976