J. Math. Soc. Japan
Vol. 50, No. 4, 1998

The dimensions of self-similar sets

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(Received Oct. 14, 1996)

1. Introduction.

Let ϕ_i be similar contraction mappings in $\boldsymbol{R}^d$ with ratios c_i, $1 \le i \le n$. Hu [5] proved that there exists unique compact set $F \subset \boldsymbol{R}^d$ such that

$$F = \bigcup_{i=1}^{n} \phi_i(F). \tag{1}$$

Further $\dim_H F = \dim_B F = \dim_P F = s$ and F is an s-set where s is such that

$$\sum_{i=1}^{n} c_i^s = 1, \tag{2}$$

if ϕ_i's satisfy the open set condition, i.e. there is a bounded nonempty open set O such that

$$\bigcup_{i=1}^{n} \phi_i(O) \subset O \tag{3}$$

with the left hand is disjoint union. Recently Sc [10] proved that F is an s-set here $\sum_{i=1}^{n} c_i^s = 1$ if and only if ϕ_i's satisfy the open condition.

Now for $\varepsilon > 0$ write

$$\Omega(\varepsilon) = \{\sigma \in S^* \,|\, c_\sigma \le \varepsilon \text{ and } c_{\sigma|(|\sigma|-1)} > \varepsilon\},$$

where $S^* = \bigcup_{i=1}^{\infty} \{1, 2, \ldots, n\}^i$ and $c_\sigma = c_{\sigma(1)} c_{\sigma(2)} \cdots c_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) \in S^*$. And for $\sigma \in S^*$, $|\sigma|$ denotes the length of σ and $\sigma|k = (\sigma(1), \ldots, \sigma(k))$ for $k \le |\sigma|$. Let $A \subset \boldsymbol{R}^d$ be a bounded open set with $A \supset F$. It is easy to see that $c_0 \varepsilon < c_\sigma \le \varepsilon$ for any $\sigma \in \Omega(\varepsilon)$ where $c_0 = \min_{1 \le i \le n} c_i$. We introduce nonnegative real numbers $\alpha_0(A)$ and $\beta_0(A)$ as follows

$$\alpha_0(A) = \sup\left\{ \alpha \,\middle|\, \underline{\lim}_{\varepsilon \to 0} \frac{\varepsilon^{-d} \mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{s(1-\alpha)}} = \infty \right\}, \tag{4}$$

$$\beta_0(A) = \sup\left\{ \beta \,\middle|\, \overline{\lim}_{\varepsilon \to 0} \frac{\varepsilon^{-d} \mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{s(1-\beta)}} = \infty \right\}, \tag{5}$$

where $\phi_\sigma = \phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) \in S^*$ and $\mathrm{m_d}(B)$ is the Lebesgue measure of $B \subset \boldsymbol{R}^d$.

Partially supported by a grant from the National Science Foundation of China. The authors' permanent address is: Dept. of Math., Central China Normal University, Wuhan 430079, P.R.China. AMS Subject Classifications: 28A80