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On 4-manifolds which admit geometric decompositions

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Introduction.

An n-manifold X is geometric in the sense of Thurston if its universal covering space $\tilde{X}$ admits a complete homogeneous Riemannian metric, $\pi_1(X)$ acts isometrically on $\tilde{X}$ and $X = \pi_1(X)\backslash\tilde{X}$ has finite volume. Every closed 1- or 2-manifold is geometric. Much current research on 3-manifolds is guided by Thurston's Geometrization Conjecture, that every closed irreducible 3-manifold admits a finite decomposition into geometric pieces [**Th82**]. There are 19 maximal 4-dimensional geometries; one of these is in fact an infinite family of closely related geometries and one is not realized by any closed 4-manifold [**F**]. Our first result (in §1) shall illustrate the limitations of geometry in higher dimensions by showing that a closed 4-manifold which admits a finite decomposition into geometric pieces is usually either geometric or aspherical. The geometric viewpoint is nevertheless of considerable interest in connection with complex surfaces [**Ue90,91**, **Wl85,86**]. We show also that except for the geometries $S^2 \times H^2$, $H^2 \times H^2$, $H^2 \times E^2$ and perhaps $\tilde{SL} \times E^1$ no closed geometric manifold has a proper geometric decomposition. In the rest of the paper we investigate connections between geometric decompositions and fibrations. In §2 we characterize algebraically the homotopy types of (orientable) Seifert fibred 4-manifolds. This class of 4-manifolds includes all but three of the 4-dimensional infrasolvmanifolds with one of the geometries E^4, Nil^4, $Nil^3 \times E^1$ or $Sol^3 \times E^1$ and all the manifolds with geometry $S^3 \times E^1$, $S^2 \times E^2$, $H^2 \times E^2$ or $\tilde{SL} \times E^1$, but also has infinitely many nongeometric members. We give examples of such manifolds which have geometric decompositions but are not geometric, and also examples which do not have geometric decompositions. In §3 we give criteria for a closed 4-manifold to be (homotopy equivalent to one which is) finitely covered by a cartesian product of closed hyperbolic surfaces. The final section determines when a 4-manifold which fibres over an aspherical closed surface with fibre a hyperbolic surface admits a geometry or a proper geometric decomposition. (Our result is incomplete in that we give only a necessary condition for such a manifold to admit one of the geometries H^4 or $H^2(C)$. There are no known examples of such bundle spaces).

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