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Lagrangian submanifolds of C^n with conformal Maslov form and the Whitney sphere

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1. Introduction

Let C^n be the n-dimensional complex Euclidean space, $\langle\ ,\ \rangle$ the Euclidean metric and J the canonical complex structure on C^n. The Kaehler two form Ω is given by $\Omega(v,w)=\langle v,Jw\rangle$, for any vectors v, w in C^n. We say that an immersion $\psi: M\to C^n$ of an n-dimensional manifold M is *Lagrangian* if $\psi^*\Omega\equiv 0$.

The simplest examples of Lagrangian submanifolds of C^n are the totally geodesic ones, i.e. the Lagrangian subspaces of C^n. A second family of examples, known as the *Whitney spheres* [**17**], can be defined as a family of Lagrangian immersions of the unit sphere S^n, centered at the origin of R^{n+1}, in C^n given by

$$\psi(x_1,\dots,x_{n+1})=\frac{r}{1+x_{n+1}^2}(x_1,x_1x_{n+1},\dots,x_n,x_nx_{n+1})+A$$

where r is a positive number and A is a vector of C^n. We will refer to r and A as the radius and the center of the Whitney sphere. Up to dilatations of C^n all the Whitney spheres are congruent with the corresponding to $r=1$, $A=0$.

These examples have very interesting properties. From a topological point of view, it is well-known that the sphere cannot be embedded in C^n as a Lagrangian submanifold [**8**]. The Whitney spheres have the best possible behaviour, because they are embedded except at the poles of S^n where they have a double point.

From the point of view of the second fundamental form, they have the simplest behaviour, after the totally geodesic ones, because their second fundamental forms σ satisfy

$$\sigma(v,w)=\frac{n}{n+2}\{\langle v,w\rangle H+\langle Jv,H\rangle Jw+\langle Jw,H\rangle Jv\},$$

being H the mean curvature vector of the Whitney sphere. In some sense, these submanifolds play the role of the umbilical hypersurfaces of the Euclidean space R^{n+1}, in the family of Lagrangian submanifolds, and it seems natural that they could be characterized of several forms, in the same way that the Euclidean spheres. The first characterization of these spheres that we get is given in Theorem 2 and says that *the totally geodesic Lagrangian submanifolds and the Whitney spheres are the only*

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