J. Math. Soc. Japan
Vol. 50, No. 1, 1998

# Characterization of the group association scheme of $A_5$ by its intersection numbers

By Masato TOMIYAMA

(Received May 12, 1995)
(Revised Nov. 9, 1995)

## 1. Introduction

Let $X$ be a finite set and let $R_i (i = 0, 1, \ldots, d)$ be relations on $X$, i.e., subsets of $X \times X$. $\mathscr{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is a *commutative association scheme of d classes* if the following conditions hold.

(1) $R_0 = \{(x, x) \mid x \in X\}$,

(2) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ and $R_i \cap R_j = \phi$ if $i \neq j$,

(3) ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x, y) \mid (y, x) \in R_i\}$,

(4) for $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant $p_{ij}^k$ whenever $(x, y) \in R_k$,

(5) $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0, 1, \ldots, d\}$.

The non-negative integers $p_{ij}^k$ are called the *intersection numbers* of $\mathscr{X}$.

An association scheme $\mathscr{X}$ is called *imprimitive* if some union of relations is an equivalence relation distinct from $R_0$ and $X \times X$, and *primitive* otherwise.

For an imprimitive association scheme $\mathscr{X}$, by rearranging the indices of the relations of $\mathscr{X}$, let $\bigcup_{i=0}^s R_i$ is an equivalence relation. For each equivalence class $X'$ of $\bigcup_{i=0}^s R_i$, we find an association scheme $\mathscr{X}' = (X', \{R_i'\}_{0 \leq i \leq s})$, where $R_i' = R_i \cap (X' \times X')$. We write $\mathscr{X} \supseteq \mathscr{X}'$. Let $\sim$ be the relation on $\{0, 1, \ldots, d\}$ defined by $i \sim j$ if and only if $p_{j,\alpha}^i \neq 0$ for some $0 \leq \alpha \leq s$. Then $\sim$ is an equivalence relation. Let $T_0 = \{0, 1, \ldots, s\}$, $T_1, \ldots, T_r$ be the equivalence classes. Then $\mathscr{X}/\mathscr{X}' = (\tilde{X}, \{\tilde{R}_i\}_{0 \leq i \leq r})$, where $X$ is the set of the equivalence classes of $\bigcup_{i=0}^s R_i$ on $X$ and $\tilde{R}_i = \{(\tilde{x}, \tilde{y}) \mid$ for $x \in \tilde{x}$ and $y \in \tilde{y}$ we have $(x, y) \in R_\alpha$ with $\alpha \in T_i\}$, is a primitive association scheme.

The reader is refered to [**2**] and [**3**] for the general theory of association schemes, and other terminology.

Let $G$ be a finite group. Let $C_0 = \{1\}, C_1, \ldots, C_d$ be the conjugacy classes of $G$. Define relations $R_i (i = 0, 1, \ldots, d)$ on $G$ by $R_i = \{(x, y) \mid yx^{-1} \in C_i\}$. Then $\mathscr{X}(G) = (G, \{R_i\}_{0 \leq i \leq d})$ is a commutative association scheme of $d$ classes, called the *group association scheme* of $G$. (See [**2**].)

It is well known that $G$ is simple if and only if the group association scheme $\mathscr{X}(G)$ is primitive.

In the study of association schemes, primitive association schemes play an important role, similar to the role simple groups play in finite groups. Namely, they are building blocks of general association schemes in the following sense. For any commutative