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On commutativity of diagrams of type II_1 factors

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1. Introduction

The subject of our study is quadruple of von Neumann algebras with inclusion relations as indicated in the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$. In addition, we assume that $R \not\subset Q$ and $Q \not\subset R$. When the von Neumann algebra K is equipped with a finite trace τ and E_Q^K, E_R^K, and E_S^K are τ-preserving conditional expectations of K onto Q, R and S, respectively, then a special situation may occur:

$$E_Q^K E_R^K = E_R^K E_Q^K = E_S^K.$$

Then the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is called a commuting square. In the special case, when $S = \boldsymbol{C}$, the subalgebras Q and R were called orthogonal by S. Popa ([**P1**]). This case, in the classical probability theory, corresponds to the condition of independence of two σ-fields.

Such diagrams were first introduced and investigated by S. Popa (cf. [**P1**], [**P2**]) and, at present, this concept is linked in a natural way with numerous problems of subfactor theory. See, for example, [**GHD**], [**K**], [**P3**], [**P4**], [**P5**], [**PP1**], [**We**], [**S**], [**Su**], [**SW**], [**WW**], [**Wi**].

Everywhere in this work, the von Neumann algebras, which form such a diagram are type II_1 factors. This situation was studied by S. Popa in [**P3**] and by T. Sano and Y. Watatani in [**SW**].

In Sections 3 and 4 we discuss sufficient conditions for the commutativity of a diagram. It may seem a little surprising that for certain inclusions, say $S \subset Q$, the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ of type II_1 factors with $[K : S] < \infty$ and $S' \cap K = \boldsymbol{C}$ must be a commuting square. We will show several results of this kind.

The notion of commuting square of type II_1 factors is strictly connected to the concept of so called co-commuting square of type II_1 factors, which will be described in

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