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Carathéodory extremal maps of ellipsoids

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1. Introduction.

Let M be a domain in $\boldsymbol{C}^n$ and $p \in M$. Let M_p denote the couple (M, p), a "pointed domain". For two pointed domains M_p and N_q, let $\mathrm{Hol}(M_p, N_q)$ denote the set of holomorphic mappings from M to N that send p to q. A map $f \in \mathrm{Hol}(M_p, N_q)$ is said to be Carathéodory extremal, or C-extremal, if

$$|\det df(p)| = \sup\{|\det dg(p)| : g \in \mathrm{Hol}(M_p, N_q)\}.$$

In case both M and N contain the origin of $\boldsymbol{C}^n$, we say f is C-extremal in $\mathrm{Hol}(M, N)$ if f is C-extremal in $\mathrm{Hol}(M_0, N_0)$. The C-extremal maps were first studied by Carathéodory in [**CAR**], and have been studied in, to name a few, [**HAR**], [**SAD**], [**KUB**], [**TRA**], [**RAB**] and [**DIT**]. In general it is very difficult or impossible to obtain explicit formulas for C-extremal maps. C-extremal maps between the ball and the polydisc were known to Carathéodory. Explicit formulas for C-extremal maps between the ball and symmetric domains were obtained by Kubota ([**KUB**]) and Travaglini ([**TRA**]). In this article we will give explicit formulas for the C-extremal maps between generalized ellipsoids (see Paragraph 2 for the definition) and the ball (Theorems 3.11 and 4.5). The generalized ellipsoids we consider are not necessarily convex. These results can be considered as an extension of the classical Schwarz lemma.

In Paragraph 2 we prove some basic properties of the C-extremal maps and of the extremal metric, and gather some known results which are needed for later paragraphs. In Paragraph 3 we give and prove the explicit formulas for the extremal maps from convex generalized ellipsoids to the ball. In Paragraph 4 we give formulas for the C-extremal maps from the ball to generalized ellipsoids, which may not be convex. In Paragraph 5 we discuss the geodesics and isolated points of the space of equivalent classes of pointed taut manifolds.

2. Basic properties.

We first give the definition of the extremal distance between pointed domains. Though we are mainly interested in bounded domains in $\boldsymbol{C}^n$, it is more natural to give the definition for complex manifolds.