J. Math. Soc. Japan
Vol. 49, No. 4, 1997

Smooth plane curves with one place at infinity

By Yuji NAKAZAWA and Mutsuo OKA

(Received Aug. 7, 1995)

1. Introduction and statement of the result.

We consider a smooth affine curve $C^a=\{f(x, y)=0\}\subset \boldsymbol{C}^2$ of degree n with one place at infinity, say at $\rho=[1;0;0]\in \boldsymbol{P}^2$ and let g be the genus of the smooth compactification of C^a. By the assumption that C^a has one place at infinity, the Newton diagram of the polynomial $f(x, y)$ has only one outside boundary and the corresponding face function has only one factor. As this place is assumed to be at ρ, $f(x, y)$ is written as

(1.1) $$f(x, y)=(y^{a_1}+\xi_1 x^{c_1})^{A_2}+(\text{lower terms}), \quad \xi_1\in \boldsymbol{C}^*, \ c_1<a_1, \ n=a_1A_2$$

where a_1, c_1, A_2 are integers and $\gcd(a_1, c_1)=1$.

If $c_1=1$, we can take the change of affine coordinates: $x'=y^{a_1}+\xi_1 x$, $y'=y$ so that the degree of $\deg f'(x', y'):=f(\xi_1^{-1}(x'-y'^{a_1}), y')$ is strictly less than n. We say C^a is *minimal* if $c_1\geqq 2$.

The purpose of this note is to classify the possible normal forms of $f(x, y)$ for a minimal smooth curve with one place at infinity of a given genus g, which we call the *generalized Abhyankar-Moh problem* or *G. A. M-problem*. Abhyankar-Moh and Suzuki independently studied the case $g=0$ ([**3**], [**12**]) and they showed that C^a is isomorphic to a line. The case $g\leqq 3$ is studied by A'Campo-Oka in [**4**] as an application of a Tschirnhausen resolution tower and we essentially follow their treatment. There also exists a work of D.W. Neumann ([**9**]) for $g\leqq 4$ from the viewpoint of the link at infinity.

Let C be the closure of C^a in $\boldsymbol{P}^2$. Recall that the homogeneous polynomial $F(X, Y, Z):=f(X/Z, Y/Z)Z^n$ defines the projective curve C in $\boldsymbol{P}^2$ and $F(X, Y, Z)$ is written as

$$F(X, Y, Z)=(Y^{a_1}+\xi_1 X^{c_1}Z^{b_1})^{A_2}+(\text{lower terms}), \quad b_1=a_1-c_1, \ n=a_1A_2$$

In the affine space $U_0:=\boldsymbol{P}^2-\{X=0\}$ with the affine coordinates $u=Z/X$, $v=Y/X$, $C\cap U_0$ is defined by $\{(u, v)\in \boldsymbol{C}^2; h(u, v)=0\}$ where

---

The second author is partially supported by Inamori foundation.