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Inner amenable semigroups 1

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1. Introduction.

Let S be a semigroup. Let $m(S)$ denote the Banach space of bounded real-valued functions on S. A *mean* on $m(S)$ is a bounded linear functional μ on $m(S)$ such that for all $f\in m(S)$, $\inf\{f(s): s\in S\}\leqq\mu(f)\leqq\sup\{f(s): s\in S\}$. An equivalent formulation is that $\|\mu\|=\mu(1)=1$, where 1 denotes the constant function on S with value 1. A mean μ on $m(S)$ is said to be a *left [respectively, right] invariant mean* if $\mu(l_s f)$ [*respectively*, $\mu(r_s f)$]$=\mu(f)$ for all $s\in S$ and $f\in m(S)$, where $l_s f$ and $r_s f$ are defined on S by $(l_s f)(t)=f(st)$ and $(r_s f)(t)=f(ts)$, $t\in S$. A mean that is both a left invariant mean (LIM) and a right invariant mean (RIM) is called an *invariant mean.* A semigroup which admits [respectively left, right] invariant means is called [respectively *left, right*] *amenable.* We refer the reader to [4] for an introductory exposition on amenable semigroups. Many results we use without mention in this article concerning amenable semigroups can be found in this reference.

We say that a mean μ on $m(S)$ is an *inner invariant mean* if $\mu(l_s f)=\mu(r_s f)$ for all $s\in S$ and $f\in m(S)$. A semigroup S which admits inner invariant means is called *inner amenable.* It follows immediately that every amenable semigroup is inner amenable. In particular, commutative semigroups are inner amenable. Another example of inner amenable semigroups is the class of semigroups with nonempty centres. Indeed, if $a\in S$ commutes with all $s\in S$, then the point mean (or Dirac measure) p_a defined by $p_a(f)=f(a)$ for all $f\in m(S)$ is an inner invariant mean on $m(S)$. We note that if S is a group, then an inner invariant mean is a mean for which $\mu(\mathfrak{T}_s f)=\mu(f)$ for all $s\in S$ and $f\in m(S)$, where $(\mathfrak{T}_s f)(t)=f(sts^{-1})=(f\circ\sigma_s)(t)$ for all $s, t\in S$, where σ_s is the inner automorphism defined by s on the group S. Inner invariant means on groups are introduced in [6] and later studied extensively by Akeman [1], H. Choda [2], M. Choda [3], Paschke [10], Pier [12], and Watatani [14], to name a few. Both Paterson [11] and Pier [13] contain an account of the study of inner amenability of groups. In this article, we shall study inner amenable semigroups and show that many classical properties concerning amenability have similar analogues for inner amenability.