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Existence of curves of genus three on a product of two elliptic curves

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1. Introduction.

Let E be an elliptic curve over the field of complex numbers, and let A be the abelian surface $E\times E$. It seems interesting to study if A contains a smooth curve of genus g. In the case when $g=2$, Hayashida and Nishi [**3**] studied this subject. Their aim was to determine if a product of two elliptic curves can be a Jacobian variety of some curve. In this note we will consider the case when $g=3$. Our first aim is to determine if A has a (1, 2)-polarization which is not a product one ([**1**]). Second one is as follows: for an algebraic variety V, the degree of irrationality $d_r(V)$ has been introduced in [**4**] or [**7**]. Especially we take an interest in the value $d_r(A)$ for an abelian surface A. Concerning this we have shown that $d_r(A)=3$ if an abelian surface A contains a smooth curve of genus 3 ([**5**]).

On the other hand the following assertion has been obtained ([**8**]):

Let n be a positive square free integer. Put $\omega=\sqrt{-n}$ [resp. $\{1+\sqrt{-n}\}/2$] if $-n\equiv 2$ or $3 \pmod 4$ [resp. $-n\equiv 1 \pmod 4$]. Let $K=\boldsymbol{Q}(\sqrt{-n})$ be an imaginary quadratic field. For each $\xi\in K\setminus\boldsymbol{Q}$, let $a\xi^2+b\xi+c=0$ be the equation of ξ satisfying that $a, b, c\in\boldsymbol{Z}$, $a>0$ and $(a, b, c)=1$. Let L be the lattice generated by $\{1, \xi\}$ and let E be the elliptic curve $\boldsymbol{C}/L$.

PROPOSITION 1. *Under the situation above, suppose that at least one of a, b, c is an even number. Then there exist two elliptic curves E_1 and E_2 on $A=E\times E$ satisfying $(E_1, E_2)=2$, where (E_1, E_2) denotes the intersection number of E_1 and E_2. Especially there exists a nonsingular curve of genus* 3 *on A, hence $d_r(A)=3$.*

REMARK 2. Of course there are many elliptic curves E satisfying the condition in this proposition. In fact, if $-n\equiv 2$ or $3 \pmod 4$, then b is even, because $a\xi$ becomes an integer. Hence every ξ enjoys the condition. For the remainder case, letting k and l $(\neq 0)$ be rational integers, we have the following.

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