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K-spherical representations for Gelfand pairs associated to solvable Lie groups

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Introduction.

Let S be a connected and simply connected unimodular solvable Lie group, K a connected compact group acting on S as automorphisms. We call the pair $(K;S)$ a Gelfand pair if the Banach $*$-algebra $L^1(K\backslash K\ltimes S/K)$ of all K-biinvariant integrable functions on $K\ltimes S$ is commutative. The assumption that $(K;S)$ is a Gelfand pair prescribes the structure of S. For example, if $(K;S)$ is a Gelfand pair, then S is of type R ([**BJR**], Corollary 7.4) and thus S has polynomial growth ([**J**], Theorem 1.4). In this paper we first show that the nilradical N of S splits in S if $(K;S)$ is a Gelfand pair. Let $\mathfrak{s}$ be the Lie algebra of S.

THEOREM A. *If $(K;S)$ is a Gelfand pair, then there exists a K-invariant abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$ on which K acts trivially. Moreover putting $A=\exp\mathfrak{a}$, one has $S=A\ltimes N$ and $K\ltimes S=(K\times A)\ltimes N$.*

Suppose that $(K;S)$ is a Gelfand pair. Since S has polynomial growth, the Banach $*$-algebra $L^1(S)$ is symmetric ([**L**], Lemma 1). This fact tells us that all bounded K-spherical functions on S are positive definite (cf. [**BJR**], Lemma 8.2). Thus to each bounded K-spherical function on S there corresponds an irreducible K-spherical representation of $K\ltimes S$ (cf. [**H**], Chapter IV). Let $\hat{N}$ be the unitary dual of N and K_π the stabilizer of $\pi\in\hat{N}$ in K. As an immediate consequence of Theorem A, we see that bounded K-spherical functions ϕ on S are parametrized as $\phi_{\pi,\alpha,a}$ with $(\pi,\alpha,a)\in\hat{N}/K\times\hat{K}_\pi\times\mathfrak{a}^*$. This parametrization improves that of [**BJR**], Theorem 8.11 a little in the sense that we make an explicit use of the subgroup $A=\exp\mathfrak{a}$.

Our second purpose of this paper is to realize the irreducible K-spherical representations $\tilde{U}_{\pi,\alpha,a}$ of $K\ltimes S$ by induction using the structure of $K\ltimes S=(K\times A)\ltimes N$, to which the parameters (π,α,a) are closely related. Though S need not be of type I, Theorem A makes it possible to get all irreducible unitary representations of $K\ltimes S$ from those of the nilradical N which is CCR. To carry out this, we need to know that N is regularly imbedded in $K\ltimes S$ and what is the structure of the stabilizers like. Our Proposition 3.2 says that the $(K\times A)$-orbit space $\hat{N}/(K\times A)$ is equal to the K-orbit space $\hat{N}/K$ as Borel spaces, which